Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau Complete Intersections in Toric Varieties

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Abstract

We formulate general conjectures about the relationship between the A-model connection on the cohomology of a d-dimensional Calabi-Yau complete intersection V of r hypersurfaces V_1, \ldots, V_r in a toric variety \mathbf{P}_{Σ} and the system of differential operators annihilating the special generalized hypergeometric function Φ_0 depending on the fan Σ . In this context, the mirror symmetry phenomenon can be interpreted as the twofold characterization of the series Φ_0 . First, Φ_0 is defined by intersection numbers of rational curves in \mathbf{P}_{Σ} with the hypersurfaces V_i and their toric degenerations. Second, Φ_0 is the power expansion near a boundary point of the moduli space of the monodromy invariant period of the homolomorphic differential *d*-form on an another Calabi-Yau *d*-fold V' called the mirror of V. Using this generalized hypergeometric series, we propose a general construction for mirrors V' of V and canonical *q*-coordinates on the moduli spaces of Calabi-Yau manifolds.

1 Introduction

In this paper we consider complex projective smooth algebraic varieties V of dimension d whose canonical bundles \mathcal{K}_V are trivial, i.e. $\mathcal{K}_V \cong \mathcal{O}_V$, and the Hodge numbers $h^{p,0}(V)$ are zero unless p = 0, or p = d. These varieties are called d-dimensional Calabi-Yau varieties, or Calabi-Yau d-folds. For each dimension $d \geq 3$, there are many of examples of topologically different Calabi-Yau d-folds which can be constructed from hypersurfaces and complete intersections in weighted projective spaces [4, 5, 6, 22].

Physicists have discovered a fascinating phenomen for Calabi-Yau manifolds, so called *mirror symmetry* [11, 16, 24, 26]. Using the mirror symmetry, Candelas et al. in [8] have computed the coefficients of the q-expansion of the Yukawa coupling for Calabi-Yau hypersurfaces of degree 5 in \mathbf{P}^4 . The method of Candelas et al. was applied to Calabi-Yau 3-folds in weighted projective spaces [13, 29, 20] and complete intersections in weighted and ordinary projective spaces [21, 25]. The q-expansions for Yukawa couplings have been calculated also for Calabi-Yau hypersurfaces of dimension d > 3 in projective spaces [17].

The interest of algebraic geometers to Yukawa couplings is explained by the conjectural relationship between the coefficients of the q-expansion of the Yukawa couplings and the intersection theory on the moduli spaces of rational curves on Calabi-Yau d-folds [17, 18]. For small values of degrees, this relationship was verified in [19]. However, the main problem which remains unsolved is to find a general rigorous mathematical explanation of this relationship for rational curves of arbitrary degrees on Calabi-Yau d-folds.

The purpose of this paper is to show that the mirror symmetry and the calculation of the Yukawa couplings for d-dimensional Calabi-Yau complete intersections in toric varieties bases essentially on the theory of special generalized hypergeometric functions. We remark that these hypergeometric functions satisfy the hypergeometric differential system considered by Gelfand, Kapranov and Zelevinsky in [14]. We propose also a general method for computing the normalized canonical q-coordinates.

The paper is organized as follows:

In Section 2, we give a review of the calculation of Candelas et al. in [8] of the coefficients Γ_d of the q-expansion of the normalized Yukawa coupling

$$K_q^{(3)} = 5 + \sum_{d \ge 1} \Gamma_d \frac{q^d}{1 - q^d}.$$

The coefficients $\Gamma_d = n_d d^3$ conjecturally coincide with the Gromov-Witten invariants (introduced by D. Morrison in [31]) for rational curves on quintic hypersurfaces in \mathbf{P}^4 . Our review is greatly influenced by the work of D. Morrison [28, 29], but we want to emphasize on the fact that the computation of the prediction for the number of rational curves on quintic 3-folds bases essentially on the properties of the special generalized hypergeometric series

$$\Phi_0(z) = \sum_{n \ge 0} \frac{(5n)!}{(n!)^5} z^n$$

which admits a combinatorial definition in terms of curves on \mathbf{P}^4 .

In Section 3, we explain a Hodge-theoretic framework for mirror symmetry and the ideas due to P. Deligne [10] and D. Morrison [30, 31] The key-point of this framework is the existence of a new-type nilpotent connection on cohomology of Calabi-Yau manifolds. Following a suggestion of D. Morrison, we call it *A*-model connection (see also [37]). The mirror symmetry identifies the A-model connection on the cohomology of a Calabi-Yau d-fold V with the classical Gauß-Manin connection on cohomology of its mirror manifold V'.

Section 4 contains a review of the standard computational technique based on the recurrent relations satisfied by coefficients of formal solutions of Picard-Fuchs equations. We use this techique later in explicit calculations of q-expansions for Yukawa couplings for some examples of Calabi-Yau complete intersections in toric varieties.

Section 5 is devoted to complete intersections in ordinary projective spaces. Using explicit description of the series $\Phi_0(z)$ for Calabi-Yau complete intersections in projective spaces, we calculate the *d*-point Yukawa coupling and propose the explicit construction for mirrors of such Calabi-Yau *d*-folds for arbitrary dimension *d*.

In Section 6, we give a general definition of special generalized hypergeometric functions and establish the relationships between these functions and combinatorial properties of rational curves on toric varieties containing Calabi-Yau complete intersections. It is easy to see that these generalized hypergeometric functions form a special subclass of the generalized hypergeometric functions with *resonance* parameters considered by Gelfand et al. in [14]. We formulate general conjectures about the differential systems and canonial q-coordinates defined by the generalized hypergeometric series corresponding to Calabi-Yau complete intersections in toric varieties. Using a combinatorial interpretation of Calabi-Yau complete intersections in toric varieties due to Yu. I. Manin, we propose an explicit construction of mirrors.

In Section 7, we consider in more details the example Calabi-Yau hypersurfaces V of degree (3,3) in $\mathbf{P}^2 \times \mathbf{P}^2$. We use this example for the illustration of the general computational method we used in Section 8, where we calculate the *q*-expansions of Yukawa couplings for some Calabi-Yau complete intersections in products of projective spaces. The computations in this section were done by the second author using an universal computer program based on Maple.

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2 Quintics in P^4

In this section we give a review of the (conjectural) computation of the Gromov-Witten invariants Γ_d and predictions n_d for numbers of rational curves of degree d on quintics V in \mathbf{P}^5 due to P. Candelas, X. de la Ossa, P.S. Green, and L. Parkes [8]. The main ingedients of this computations were considered in papers of D. Morrison [28, 29]. The purpose of this review is to stress that this computation needs knowing only properties of the special generalized hypergeometric function $\Phi_0(z)$. We begin with the computational algorithm for computing the coefficients in the q-expansion of the Yukawa coupling and the predictions for number of rational curves.

2.1 The coefficients in the q-expansion of the Yukawa coupling

Consider the series

$$\Phi_0(z) = \sum_{n \ge 0} \frac{(5n)!}{(n!)^5} z^n.$$

Step 1. If we put $a_n = \frac{(5n)!}{(n!)^5}$, then the numbers a_n satisfy the recurrent relation

$$(n+1)^4 a_{n+1} = 5(5n+1)(5n+2)(5n+3)(5n+4)a_n$$

This immediately implies that the series $\Phi_0(z)$ is the solution to the differential equation

 $\mathcal{D}\Phi(z) = 0,$

where

$$\mathcal{D} = \Theta^4 - 5z(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4), \ \Theta = z\frac{\partial}{\partial z}$$

One can rewrite the differential operator \mathcal{D} in powers of Θ as follows:

 $\mathcal{D} = A_4(z)\Theta^4 + A_3(z)\Theta^3 + \dots + A_0(z).$

We denote by $C_i(z)$ the rational function $A_i(z)/A_4(z)$ (i = 0, ..., 3).

Step 2. Following [28], define the normalized Yukawa 3-differential as

$$\mathcal{W}_3 = K_z^{(3)} (\frac{dz}{z})^{\otimes 3},$$

where $K_z^{(3)} = W_3(z)/\Phi_0^2(z)$ is the 3-point coupling function such that $W_3(z)$ satisfies the differential equation

$$\Theta W_3(z) = -\frac{1}{2}C_3(z)W_3(z)(10)$$

and the normalizing condition $W_3(0) = 5$.

One easily obtains

$$\mathcal{W}_3 = \frac{5}{(1 - 5^5 z)\Phi_0^2(z)} (\frac{dz}{z})^{\otimes 3}.$$

Step 3. The equation $\mathcal{D}\Phi = 0$ is a Picard-Fuchs differential equation with maximal unipotent monodromy (in the sense of Morrison [28]) at z = 0. Therefore, there exists a unique solution $\Phi_1(z)$ to $\mathcal{D}\Phi = 0$ such that $\Phi_1(z) = (\log z)\Phi_0(z) + \Psi(z)$, where $\Psi(z)$ is regular at z = 0 and $\Psi(0) = 0$. We define the new local coordinate q = q(z) near the point z = 0 as

$$q(z) = \exp\left(\frac{\Phi_1(z)}{\Phi_0(z)}\right) = z \exp\left(\frac{\Psi(z)}{\Phi_0(z)}\right).$$

The, we rewrite the normalized Yukawa 3-differential \mathcal{W}_3 in the coordinate q as

$$\mathcal{W}_3 = K_q^{(3)} (\frac{dq}{q})^{\otimes 3}.$$

The function $K_q^{(3)}$ is called the Yukawa 3-point coupling. This function has the power expansion

$$K_q^{(3)} = 5 + \sum_{d\ge 1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d},$$

where $\Gamma_d = n_d d^3$ are conjectured to be the Gromov-Witten invariants of rational curves of degree d on a quintic 3-fold in \mathbf{P}^4 [19, 31]. The numbers n_d are predictions for numbers of rational curves of degree d on quintic 3-folds.

It is important to remark that in the above algorithm for calculation of the numbers n_d one needs to know only properties of the series $\Phi_0(z)$ and the normalization condition $W_3(0) = \deg V = 5$ for $W_3(z)$, i.e., one does not need to know anything about mirrors of quintics.

2.2 Philosophy of mirrors and the series $\Phi_0(z)$

The central role in the computation of Candelas et al. in [8] is played by the orbifold construction of mirrors for quintics in \mathbf{P}^4 [16]. In [1], this construction of mirrors was generalized for hypersurfaces in toric Fano varieties with Gorenstein singularities.

In the above algorithm, we have shown that one can forget about mirrors. However, the philosophy of mirrors proves to be very helpful. For quintic 3-folds this philosophy appears as the following twofold interpretation of the series $\Phi_0(z)$.

The first interpretation: We compute the coefficients a_n of the power series $\Phi_0(z)$ using combinatorial properties of curves $C \subset \mathbf{P}^4$ of degree n.

Notice that any such a curve C meets a generic quintic V at 5n distinct points p_1 , ..., p_{5n} . There exists a degeneration of V into a union of 5 hyperplanes $H_1 \cup \cdots \cup H_5$. Every such a hyperplane H_i intersects C at n points p_{i_1}, \ldots, p_{i_n} which can be considered as deformations of a subset of n points from the set $\{p_1, \ldots, p_{5n}\}$. It remains to remark that there exists exactly $(5n)!/(n!)^5$ ways to divide $\{p_1, \ldots, p_{5n}\}$ into 5 copies of n-element disjoint subsets.

The second interpretation: We find the coefficients a_n from an integral representation of $\Phi_0(z)$.

Let $\mathbf{T} \cong (\mathbf{C}^*)^4$ be the 4-dimensional algebraic torus with coordinate functions X_1, X_2, X_3, X_4 . Take the Laurent polynomial

$$f(u,X) = 1 - (u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4 + u_5(X_1X_2X_3X_4)^{-1})$$

in variables X_1, X_2, X_3, X_4 , where the coefficients u_1, \ldots, u_5 are considered as independent parameters. Let $z = u_1 u_2 u_3 u_4 u_5$.

Proposition 2.2.1

$$\Phi_0(u_1\cdots u_5) = \Phi_0(z) = \frac{1}{(2\pi\sqrt{-1})^4} \int_{|X_i|=1} \frac{1}{f(u,X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}.$$

Proof. One has

$$\frac{1}{f(u,X)} = \sum_{n \ge 0} (u_1 X_1 + u_2 X_2 + u_3 X_3 + u_4 X_4 + u_5 (X_1 X_2 X_3 X_4)^{-1})^n$$
$$= \sum_{m \in \mathbf{Z}^4} c_m(u) X^m.$$

It is straightforward to see that $c_0(u) = \Phi_0(u_1 \cdots u_5)$. Now the statement follows from the Cauchy residue formula.

The second interpretation of $\Phi_0(z)$ implicitly uses mirrors of quintics, since zeros of f(u, X) define the affine Calabi-Yau 3-fold Z_f in **T** whose smooth Calabi-Yau compactification is mirror symmetric with respect to quintic 3-folds (see [1]). Moreover, the holomorphic 3-form ω on Z_f such that $\omega(z)$ extends regularly to a smooth compactification of Z_f depends only on z, i.e., only on the product $u_1 \cdots u_5$. This 3-form can be written as

$$\omega(z) = \frac{1}{(2\pi\sqrt{-1})^4} \operatorname{Res} \frac{1}{f(u,X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}.$$

This shows that $\Phi_0(z)$ is exactly the monodromy invariant period of the 3-form $\omega(z)$ near z = 0.

Proposition 2.2.2 The differential 3-form $\omega(z)$ satisfies the same Picard-Fuchs differential equation $\mathcal{D}\Phi = 0$ as for the series $\Phi_0(z)$. In particular, all periods of $\omega(z)$ satisfy the Picard-Fuch differential equation with the operator

$$\Theta^4 - 5z(5\Theta + 1)(5\Theta + 2)(5\Theta + 3)(5\Theta + 4).$$

Proof. In order to prove the statement, it is sufficient to check that

$$\left(\mathcal{D}\frac{1}{f(u,X)}\right)\frac{dX_1}{X_1}\wedge\frac{dX_2}{X_2}\wedge\frac{dX_3}{X_3}\wedge\frac{dX_4}{X_4}$$

is a differential of a rational 3-form on $\mathbf{T} \setminus Z_f$. The latter follows from standard arguments using reduction by the Jacobian ideal J_f (see [2]).

2.3 A-model connection

The Yukawa coupling can be described by a nilpotent connection ∇_A on the cohomology of quintic 3-fold V

$$\nabla_A : H^*(V, \mathbf{C}) \to H^*(V, \mathbf{C}) \otimes \mathbf{C} \langle \frac{dz}{z} \rangle.$$

This connection is homogeneous of degree 2, i.e.,

$$\nabla_A : H^i(V, \mathbf{C}) \to H^{i+2}(V, \mathbf{C}) \otimes \mathbf{C} \langle \frac{dz}{z} \rangle$$

and hence ∇_A vanishes on $H^3(V, \mathbb{C})$. By this reason, we consider only the cohomology subring

$$H^{2*}(V, \mathbf{Z}) = \bigoplus_{i=0}^{3} H^{2i}(V, \mathbf{Z}) \subset H^*(V, \mathbf{Z})$$

of even-dimensional classes on a quintic 3-fold V (rk $H^{2i}(V, \mathbf{Z}) = 1$). Let η_i be the positive generator of $H^{2i}(V, \mathbf{Z})$. Then, in the basis $\eta_0, \eta_1, \eta_2, \eta_3$, the multiplication by η_1 is the endomorphism of $H^{2*}(V, \mathbf{Z})$ having as matrix

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Following [9] and [31], we define the 1-parameter connection on $H^{2*}(V, \mathbb{C}) \otimes \mathbb{C}[[q]]$ considered as a trivial bundle over Spec $\mathbb{C}[[q]]$ as follows

can be considered as the deformation of the matrix Λ such that

$$\Lambda = \operatorname{Res}|_{q=0} K(q).$$

The mirror philosophy shows that the matrix (1) can be identified with the matrix of the classical Gauß-Manin connection on the 4-dimensional cohomology space $H^3(\hat{Z}_f, \mathbf{C})$ in a special symplectic basis. We notice that the quotients F^i/F^{i+1} of the Hodge filtration

$$H^3(\hat{Z}_f, \mathbf{C}) = F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4 = 0$$

are 1-dimensional. There is also the monodromy filtration on the homology $H_3(\hat{Z}_f, \mathbf{Z})$

$$0 = W_{-4} \subset W_{-3} \subset W_{-2} \subset W_{-1} \subset W_0 = H_3(\hat{Z}_f, \mathbf{Z})$$

such that W_i/W_{i-1} are also 1-dimensional. We choose the symplectic basis $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $H_3(\hat{Z}_f, \mathbf{Z})$ in such a way that $\{\gamma_0, \ldots, \gamma_i\}$ form a **Z**-basis of W_i . We choose also the basis $\omega_0, \omega_1, \omega_2, \omega_3$ of $H^3(\hat{Z}_f, \mathbf{C})$ such that $\{\omega_0, \ldots, \omega_i\}$ form a **C**-basis of F^{3-i} and

$$p_{ij} = \int_{\gamma_j} \omega_i = \delta_{ij} \text{ for } i \ge j$$

So the period matrix $\Pi = (p_{ij})$ has the form [17, 31]

$$\Pi = \begin{pmatrix} 1 & p_{12} & p_{13} & p_{14} \\ 0 & 1 & p_{23} & p_{34} \\ 0 & 0 & 1 & p_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that all coefficients p_{ij} (i < j) are multivalued functions of z near z = 0.

Applying the Griffiths transversality property, we obtain that the Gauß–Manin connection in the z-coordinate has the form

$$\begin{pmatrix} \nabla \omega_0 \\ \nabla \omega_1 \\ \nabla \omega_2 \\ \nabla \omega_3 \end{pmatrix} = \begin{pmatrix} 0 & (\Theta p_{12}) \frac{dz}{z} & 0 & 0 \\ 0 & 0 & (\Theta p_{23}) \frac{dz}{z} & 0 \\ 0 & 0 & 0 & (\Theta p_{34}) \frac{dz}{z} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix},$$

where $\Theta p_{i,i+1}$ are single valued functions.

Then the Yukawa 3-differential is simply the tensor product

$$\mathcal{W}_3 = K_z^{(3)} (\frac{dz}{z})^{\otimes 3} = (\Theta p_{12}) \frac{dz}{z} \otimes (\Theta p_{23}) \frac{dz}{z} \otimes (\Theta p_{34}) \frac{dz}{z}.$$

By Griffiths transversality, one has $\omega_0 \wedge \omega_2 = 0$, i.e. we can assume that $p_{23} = p_{12}$. The differential form w_0 can be defined as $\omega/\Phi_0(z)$. Moreover, $p_{12} = \Phi_1(z)/\Phi_0(z)$. In the new coordinate q, we have $p_{12} = \log q$. Then the Gauß-Manin connection can be rewritten as

$$\begin{pmatrix} \nabla\omega_0\\ \nabla\omega_1\\ \nabla\omega_2\\ \nabla\omega_2\\ \nabla\omega_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{dq}{q} & 0 & 0\\ 0 & 0 & K_q^{(3)}\frac{dq}{q} & 0\\ 0 & 0 & 0 & \frac{dq}{q}\\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_0\\ \omega_1\\ \omega_2\\ \omega_3 \end{pmatrix}$$

The role played by the 1-form dq/q in the connection matrix was first observed by P. Deligne [10].

2.4 The *q*-coordinate and the Yukawa coupling

Since the coordinate q was defined intrinsically as the ratio $\Phi_1(z)/\Phi_0(z)$ of two solutions of the differential equation $\mathcal{D}\Phi = 0$, it is natural to ask about the form of the differential operator \mathcal{D} in the new coordinate q. Denote by Ξ the differential operator $q\frac{\partial}{\partial q}$.

Proposition 2.4.1 The differential 3-form ω_0 satisfies the Picard-Fuchs differential equation with the differential operator

$$\Xi^4 + c_3(q)\Xi^3 + c_2(q)\Xi^2$$
,

where

$$c_3(q) = -2 \frac{\Xi K_q^{(3)}}{K_q^{(3)}}, \ c_2(q) = \frac{\Xi K_q^{(3)}}{(K_q^{(3)})^2} - \frac{\Xi^2 K_q^{(3)}}{K_q^{(3)}}.$$

Proof. By properties of the nilpotent connection, one has

$$\Xi\omega_0 = \omega_1, \ \Xi\omega_1 = K_q^{(3)}\omega_2, \ \Xi\omega_2 = \omega_3, \ \Xi\omega_3 = 0.$$

 So

$$\Xi^{4}\omega_{0} = \Xi^{2}K_{q}^{(3)}\omega_{2} = \Xi((\Xi K_{q}^{(3)})\omega_{2} + K_{q}^{(3)}\omega_{3})$$
$$= (\Xi^{2}K_{q}^{(3)})\omega_{2} + 2(\Xi K_{q}^{(3)})\omega_{3}.$$

On the other hand,

$$\omega_2 = \frac{1}{K_q^{(3)}} \Xi^2 \omega_0,$$

$$\omega_3 = \Xi \left(\frac{1}{K_q^{(3)}} \Xi^2 \omega_0\right) = -\frac{\Xi K_q^{(3)}}{(K_q^{(3)})^2} \Xi^2 \omega + \frac{1}{K_q^{(3)}} \Xi^3 \omega_0.$$

Remark 2.4.2 The differential equation for ω_0 can be written also as

$$\Xi^2 (K_q^{(3)})^{-1} \Xi^2 \omega_0 = 0.$$

In this form this equation first arose in [12].

The differential operator \mathcal{D} which annihilates the function $\Phi_0(z)$ defines the connection in the basis ω , $\Theta\omega$, $\Theta^2\omega$, $\Theta^3\omega$ of $H^3(\hat{Z}_f, \mathbb{C})$:

$$\begin{pmatrix} \nabla \omega \\ \nabla \Theta \omega \\ \nabla \Theta^2 \omega \\ \nabla \Theta^3 \omega \end{pmatrix} = \begin{pmatrix} 0 & \frac{dz}{z} & 0 & 0 \\ 0 & 0 & \frac{dz}{z} & 0 \\ 0 & 0 & 0 & \frac{dz}{z} \\ -C_0(z)\frac{dz}{z} & -C_1(z)\frac{dz}{z} & -C_2(z)\frac{dz}{z} & -C_3(z)\frac{dz}{z} \end{pmatrix} \begin{pmatrix} \omega \\ \Theta \omega \\ \Theta^2 \omega \\ \Theta^3 \omega \end{pmatrix}.$$

The basis ω , $\Theta\omega$, $\Theta^2\omega$, $\Theta^3\omega$ is also compatible with the Hodge filtration in $H^3(\hat{Z}_f, \mathbb{C})$. Thus there exist a matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{34} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{pmatrix}$$

such that

$$\begin{pmatrix} \omega \\ \Theta \omega \\ \Theta^2 \omega \\ \Theta^3 \omega \end{pmatrix} = R \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

It is easy to see that

$$r_{11} = \Phi_0(z), \ r_{22} = \Phi_0(z)(\Theta p_{12}), \ r_{33} = \Phi_0(z)(\Theta p_{12})(\Theta p_{23}),$$
$$r_{44} = \Phi_0(z)(\Theta p_{12})(\Theta p_{23})(\Theta p_{34}).$$

3 Quantum variations of Hodge structure on Calabi-Yau manifolds

3.1 A-model connection and rational curves

A general appoach to the definition of a new connection on cohomology of algebraic and symplectic manifolds V was proposed by Witten [36]. The construction of Witten bases on the interpretation of third partial derivatives

$$\frac{\partial^3}{\partial z_i \partial z_j \partial z_k} P(z)$$

of a function P(z) on the cohomology space $H^*(V, \mathbb{C})$ as structure constants of a commutative associative algebra. The function P(z) is defined via the intersection theory on the moduli spaces of mappings of Riemann surfaces S to V. Using Poincare duality, one obtains the structure coefficients of the connection on $H^*(V, \mathbb{C})$.

We consider a specialization of the general construction to the case when V is a Calabi-Yau 3-fold. We put $n = \dim H^2(V, \mathbf{C}) = \dim H^4(V, \mathbf{C})$. Let η_0 be a generator of $H^0(V, \mathbf{Z})$, η_1, \ldots, η_n a **Z**-basis of $H^2(V, \mathbf{Z}), \zeta_1, \ldots, \zeta_n$ the dual **Z**-basis of $H^4(V, \mathbf{Z})$ ($\langle \eta_i, \zeta_j \rangle = \delta_{ij}$), and ζ_0 the dual to η_0 generator of $H^6(V, \mathbf{Z})$. We can always assume that the cohomology classes η_1, \ldots, η_n are contained in the closed Kähler cone of V.

Definition 3.1.1 Let $R = \mathbf{C}[[q_1, \ldots, q_n]]$ be the ring of formal power series in n independent variables. We denote by H(V) the scalar extension

$$(\oplus_{i=0}^{3} H^{2i}(V, \mathbf{C})) \otimes_{\mathbf{C}} R.$$

We consider a flat nilpotent holomorphic connection

$$\nabla_A : H(V) \to H(V) \otimes \Omega^1_R(\log q)$$

defined by the following formulas [9, 31]

$$abla_A \eta_0 = \sum_{i=1}^n \eta_i \otimes \frac{dq_i}{q_i};$$

$$\nabla_A \eta_k = \sum_{i=1}^n \sum_{j=1}^n K_{ijk} \zeta_j \otimes \frac{dq_i}{q_i}, \ k = 1, \dots, n;$$

$$\nabla_A \zeta_j = \zeta_0 \frac{dq_j}{q_j}, \ j = 1, \dots, n;$$

 $\nabla_A \zeta_0 = 0.$

The coefficients K_{ijk} are power series in q_1, \ldots, q_n defined by rational curves C on V, i.e., morphisms $f : \mathbf{P}^1 \to V$ as follows

$$K_{ijk} = \langle \eta_i, \eta_j, \eta_k \rangle + \sum_{\substack{C \subset V \\ [C] \neq 0}} n_{[C]} \langle C, \eta_i \rangle \langle C, \eta_j \rangle \langle C, \eta_k \rangle \frac{q^{[C]}}{1 - q^{[C]}},$$

where $q^{[C]} = q_1^{c_1} \cdots q_n^{c_n} (c_i = \langle C, \eta_i \rangle)$. The integer

$$\Gamma_{[C]}(\eta_i, \eta_j, \eta_k) = n_{[C]} \langle C, \eta_i \rangle \langle C, \eta_j \rangle \langle C, \eta_k \rangle$$

is called the Gromov-Witten invariant [19, 31] of the class [C]. If the classes η_i , η_j , and η_k are represented by effective divisors D_i , D_j , and D_k on V, then $\Gamma_{[C]}(\eta_i, \eta_j, \eta_k)$ is the number of pseudo holomorphic immersions $i : \mathbf{P}^1 \to V$ such that $[i(\mathbf{P}^1)] = [C]$ and $i(0) \in D_i$, $i(1) \in D_j, i(\infty) \in D_k$ for sufficiently general almost complex structure on V. In particular, the number $n_{[C]}$ which predicts the number of rational curves $C \subset V$ with the fixed class [C] is always non-negative.

The connection ∇_A will be called the *A*-model connection. The connection ∇_A defines on H(V) a variation of Hodge structure of type (1, n, n, 1). We call this variation the quantum variation of Hodge structure on V.

Remark 3.1.2 The Picard-Fuch differential system satisfied by η_0 was considered in details in [9].

One immediately obtains:

Proposition 3.1.3 Let $\eta = l_1\eta_1 + \cdots + l_n\eta_n \in H^2(V, \mathbb{Z})$ be a class of an ample divisor on V. Define the 1-parameter connection with the new coordinate q by putting $q_1 = q^{l_1} \dots, q_n = q^{l_n}$. Then the connection ∇_A on H(V) induces the connection

$$\nabla_q : (\bigoplus_{i=0}^3 H^{2i}(V, \mathbf{C}[[q]])) \to (\bigoplus_{i=0}^3 H^{2i}(V, \mathbf{C}[[q]])) \otimes_{\mathbf{C}} \Omega^1_{\mathbf{C}[[q]]}(\log q).$$

In particular, the residue of the connection operator ∇_q at q = 0 is the Lefschetz operator $L_\eta : H^{2i}(V, \mathbf{C}) \to H^{2i+2}(V, \mathbf{C})$, and

$$\left\langle (\nabla_q)^3 \eta_0, \zeta_0 \right\rangle = \left(\left\langle \eta, \eta, \eta \right\rangle + \sum_{d>0} n_d \frac{d^3 q^d}{1 - q^d} \right) \left(\frac{dq}{q} \right)^{\otimes 3}$$

where

$$n_d = \sum_{\langle C, \eta \rangle = d} n_{[C]}.$$

Corollary 3.1.4 The connection ∇_q defines a Picard-Fuchs differential operator of order 4 annihilating η_0 .

3.2 The Gauß-Manin connection for mirrors

Let W be a Calabi-Yau 3-fold such that $\dim H^3(W, \mathbb{C}) = 2n + 2$. Assume that we are given a variation W_z of complex structure on W near a boundary point p of the *n*-dimensional moduli space \mathcal{M}_W of complex structures on W in holomorphic coordinates z_1, \ldots, z_n near p such that $p = (0, \ldots, 0)$.

Definition 3.2.1 The family W_z is said to have the maximal unipotent monodromy at z = 0 if the weight filtration

$$0 = W_{-1} \subset W_0 \subset W_1 \subset W_2 \subset W_3 = H^3(W_z, \mathbf{C})$$

defined by N is orthogonal to the Hodge filtration $\{F^i\}$, i.e.,

$$H^{3}(W_{z}, \mathbf{C}) = W_{i}^{\perp} \oplus F^{3-i} \ i = 0, \dots, 3.$$

(This is essentially the same definition given in [30, 31].)

Choose a symplectic basis

$$\gamma_0, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots \delta_n, \delta_0$$

of $H_3(W_z, \mathbf{Z})$ in such a way that γ_0 generates $W_0, \gamma_0, \gamma_1, \ldots, \gamma_n$ is a **Z**-basis of W_1 ,

$$\gamma_0, \gamma_1, \ldots, \gamma_n, \delta_1, \ldots \delta_n$$

is a **Z**-basis of W_2 . Then we choose a symplectic basis in $H^3(W_z, \mathbb{C})$:

$$\omega_0, \omega_1, \ldots, \omega_n, \nu_1, \ldots, \nu_n, \nu_0$$

such that ω_0 generates F^3 , $\omega_0, \omega_1, \ldots, \omega_n$ is the basis of F^2 , $\omega_0, \omega_1, \ldots, \omega_n, \nu_1, \ldots, \nu_n$ is the basis of F^1 such that

$$\langle \omega_i, \gamma_i \rangle = \langle \nu_i, \delta_i \rangle = 1, \ i = 0, 1, \dots, n$$

$$\langle \omega_i, \gamma_0 \rangle = \langle \nu_j, \gamma_0 \rangle = \langle \nu_i, \gamma_j \rangle = \langle \nu_0, \gamma_j \rangle = \langle \nu_0, \delta_j \rangle = 0, \ i = 1, \dots, n, \ j = 0, \dots, n.$$

The choice of the basis of $H^3(W_z, \mathbb{C})$ defines the splitting into the direct sum

$$H^{3}(W_{z}, \mathbf{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

such that all direct summand acquire canonical integral structures. By Griffiths transversality property, the Gauß-Manin connection ∇ sends $H^{3-i,i}$ to $H^{3-i-1,i+1} \otimes \Omega^1(\log z)$. It is an observation of Deligne [10] that the weight and Hodge filtrations define a variation of mixed Hodge structure (VMHS).

Two Calabi-Yau 3-folds V and W are called mirror symmetric if the quantum variation of Hodge structure for V is isomorphic to the classical VMHS for W. In this case the q-coordinates near p up to constants are defined by the formula [30]

$$q_i = \exp(2\pi\sqrt{-1}) \int_{\gamma_i} \omega_0.$$

4 Picard-Fuchs equations

In this section we recall standard facts about Picard-Fuch differential equations which we use in computations of Yukawa *d*-point functions and predictions for numbers of rational curves on Calabi-Yau manifolds.

4.1 Recurrent relations and differential equations

Let a_n n = 0, 1, 2, ... be an infinite sequence of complex numbers. For our purposes, it will be more convenient to define a_n for all integers $n \in \mathbb{Z}$ by putting $a_n = 0$ for n < 0. We define the generating function for the sequence $\{a_i\}$ as the formal power series

$$\Phi(z) = \sum_{i \ge 0}^{\infty} a_i z^i \in \mathbf{C}[[z]]$$

Consider two differential operators acting on $\mathbf{C}[[z]]$:

$$\Theta : f \mapsto z \frac{\partial}{\partial z} f,$$
$$z : f \mapsto z \cdot f$$

satisfying the relation

$$[\Theta, z] = \Theta \circ z - z \circ \Theta = z.$$
⁽²⁾

These operators generate the algebra $\mathbf{D} = \mathbf{C}[z, \Theta]$ of "logarithmic" differential operators which are polynomials in non-commuting operators Θ and z

Fix a positive integer d. Assume that there exist m+1 $(m \ge 1)$ polynomials

 $P_0(y),\ldots,P_m(y)\in \mathbf{C}[y]$

of degree d + 1 such that for every $n \in \mathbb{Z}$ the numbers $\{a_i\}$ satisfy the recurrent relation:

$$P_0(n)a_n + P_1(n+1)a_{n+1} + \dots + P_m(n+m)a_{n+m} = 0.$$
(3)

(Here we consider y as a new complex variable having no connection to our previous variable z.) Then $\Phi(z)$ is a formal solution of the linear differential equation

$$\mathcal{D}\Phi(z) = 0$$

with the differential operator

$$\mathcal{D} = z^m P_0(\Theta) + z^{m-1} P_1(\Theta) + \dots + P_m(\Theta).$$
(4)

This differential equation of order d + 1 can be rewritten in powers of Θ as

$$\mathcal{D} = A_{d+1}(z)\Theta^{d+1} + \dots + A_1(z)\Theta + A_0(z), \tag{5}$$

where A_i are some polynomials in z. It is easy to check the following:

Proposition 4.1.1 A power series $\Phi(z)$ is a formal solution to a differential equation $\mathcal{D}\Phi(z) = 0$ of order d + 1 for some element $\mathcal{D} \in \mathbf{D}$ if and only if the coefficients $\{a_i\}$ satisfy a recurrent relation as in (3) for some polynomials $P_0(y), \ldots, P_m(y)$ of degree d + 1.

4.2 Picard-Fuchs operators

Recall that a differential operator \mathcal{D} as in (5) is called a *Picard-Fuchs operator at* point z = 0 if $A_{d+1}(0) \neq 0$. Moreover, the condition that solutions of the Picard-Fuchs equatios $\mathcal{D}\Phi$ have maximal unipotent monodromy at z = 0 [29] is equivalent to $A_i(0) = 0$ for $i = 0, \ldots, d$. These properties of the operator \mathcal{D} can be reformulated in terms of the properties of the polynomial $P_m(y)$ in (3) as follows:

A differential operator \mathcal{D} is a Picard-Fuchs operator if and only if the polynomial $P_m(y)$ has degree d+1, i.e., its leading coefficient is nonzero. Moreover, solutions of the equations \mathcal{D} have maximal unipotent monodromy at z = 0 if and only if the polynomial $P_m(y)$ equals cy^k for some nonzero constant c.

Picard-Fuchs operators having the maximal unipotent monodromy at z = 0 will be objects of our main interest. Therefore, we introduce the following definition:

Definition 4.2.1 A Picard-Fuchs operator \mathcal{D} with the maximal unipotent monodromy will be called a *MU-operator*. We will always assume that the corresponding polynomial $P_m(y)$ in (3) for any *MU*-operator \mathcal{D} is y^k , i.e., c = 1.

The fundamental property of MU-operators is the following one:

Theorem 4.2.2 If \mathcal{D} is MU-operator, then the subspace in $\mathbf{C}[[z]]$ of solutions of the linear differential equation

$$\mathcal{D}\Phi(z) = 0$$

has dimension 1. Moreover, every solution is defined uniquely by the value $\Phi(0) = a_0$.

Proof. If we have chosen a value of a_0 , the all coefficients a_i for $i \ge 0$ are uniquely defined from the recurrent relation (3). (We remind, that we put $a_i = 0$ for i < 0.) \Box

Definition 4.2.3 Let \mathcal{D} be a *MU*-operator. Then the power series solution $\Phi_0(z)$ of the equation $\mathcal{D}\Phi(z) = 0$ normalized by the condition $\Phi_0(0) = 1$ will be called the *socle-solution*.

4.3 Logarithmic solutions and the *q*-coordinate

Let \mathcal{D} be a *MU*-operator of order d+1. Putting $C_i(z) = A_i(z)/A_{d+1}(z)$ we can define another differential operator

$$\mathcal{P} = \mathcal{P}(\Theta) = \sum_{i=0}^{d+1} C_i(z)\Theta^i$$

which is also a MU-operator of order d + 1, where $C_i(z)$ are rational functions in z, and $C_{d+1}(z) \equiv 1$. Assume that we have a formal regular solution

$$\Phi(z) = \sum_{i=0}^{\infty} a_n z^n.$$

Consider a formal polynomial extension

$$M_z = \mathbf{C}[[z]][\log z],$$

where $\log z$ is considered as a new transcendent variable. We can define the structure of a left **D**-module on M_z putting by definition $\Theta \log z = 1$. In fact, M_z will be a module over the larger algebra \mathbf{D}_z containing the new operator Log z such that

$$z \circ (\Theta \circ Logz) = (\Theta \circ Log z) \circ z = 1,$$
$$\Theta \circ Log z - Log z \circ \Theta = 1,$$

and Log z acts on M_z by multiplication on $\log z$.

Proposition 4.3.1 Let $\mathcal{P} = \sum_{i=0}^{d+1} C_i(z) \Theta^i$ be any operator in **D**. Then

$$[\mathcal{P}, Log \, z] = \sum_{i=1}^{d+1} iC_i(z)\Theta^{i-1} = \mathcal{P}'_{\Theta},$$

where \mathcal{P}'_{Θ} is a formal derivative of \mathcal{P} with respect to Θ .

Proof. The statement follows from relation

$$\Theta^i \circ Log \, z - Log \, z \circ \Theta^i = i \Theta^{i-1}$$

which can be proved by induction.

Assume that we want to find a element $\Phi_1(z)$ in M_z such that $\mathcal{P}\Phi_1(z) = 0$ and $\Phi_1(z)$ has form

$$\Phi_1(z) = \log z \cdot \Phi_0(z) + \Psi(z)$$

where $\Psi(z)$ is an element of $\mathbf{C}[[z]]$, and $\Psi(0) = 0$.

Proposition 4.3.2 The element $\Psi(z)$ satisfies the linear non-homogeneous differential equation

$$\mathcal{P}_{\Theta}^{\prime}\Phi_{0}(z) + \mathcal{P}\Psi(z) = 0, \tag{6}$$

or, formally,

$$\Psi(z) = -\mathcal{P}^{-1}\mathcal{P}'_{\Theta}\Phi_0(z) = \partial_{\Theta}\log\mathcal{P}\cdot\Phi_0(z).$$

Proof. Since Φ_0 and Φ_1 are solutions, we obtain

$$0 = \mathcal{P}\Phi_1 = \mathcal{P}\log z\Phi_0 + \mathcal{P}\Psi =$$

 $=(Log \ z \circ \mathcal{P}+[\mathcal{P},Log \ z])\Phi_0+\mathcal{P}\Psi=[\mathcal{P},Log \ z]\circ\Phi_0(z)+\mathcal{P}\Psi=\mathcal{P}'_{\Theta}\Phi_0+\mathcal{P}\Psi_0=0.\ \Box$

Proposition 4.3.3 If $\Phi_0(z)$ is the socle solution, then the function $\Psi(z)$ is uniquely defined by the equations (6) and the condition $\Psi(0) = 0$ as an element of $\mathbf{C}[[z]]$.

Proof. Let $\Psi(z) = \sum_{i=-\infty}^{+\infty} b_i z^i$ be an element of $z\mathbf{C}[[z]]$, i.e., $b_i = 0$ for $i \leq 0$. By 4.3.2, for any $n \in \mathbf{Z}$, the coefficient by z^n in $\mathcal{P}\Psi(z)$ is

$$P_m(n)b_n + P_{m-1}(n-1)b_{n-1} + \dots + P_0(n-m)b_{n-m}$$

On the other hand, the coefficient by z^n in $\mathcal{P}'_{\Theta}\Phi_0(z)$ is

$$P'_{m}(n)a_{n} + P'_{m-1}(n-1)a_{n-1} + \dots + P'_{0}(n-m)a_{n-m}$$

Thus, we obtain the recurrent linear non-homogeneous relation

$$P_m(n)b_n + \sum_{i=1}^m P_{m-i}(n-i)b_{n-i} + \sum_{i=0}^m P'_{m-i}(n-i)a_{n-i} = 0.$$
(7)

Since $P_m(n) = n^{d+1} \neq 0$ for $n \ge 1$, one can find all coefficients b_i $(i \ge 1)$ using (7). For instance, we obtain

$$b_1 = -(d+1)a_1 - P'_{m-1}(0)a_0$$

$$2^{d+1}b_2 = -(d+1)2^d a_2 - P_{m-1}(1)b_1 - P'_{m-1}(1)a_1 - P'_{m-2}(0)a_0; \dots \text{ etc.}$$

Corollary 4.3.4 Let \mathcal{P} be a MU-operator then the quotient Ψ/Φ of the solutions of the linear system

$$\mathcal{P}\Phi = 0, \ \mathcal{P}'_{\Theta}\Phi + \mathcal{P}\Psi, \ \Psi(0) = 0$$

is a function depending only on \mathcal{P} .

We come now to the most important definition:

Definition 4.3.5 The element

$$q = \exp\left(\frac{\Phi_1(z)}{\Phi_0(z)}\right) = z \exp\left(\frac{\Psi(z)}{\Phi_0(z)}\right)$$

is called the *q*-parameter for the MU-operator \mathcal{P} .

4.4 Generalized hypergeometric functions and 2-term recurrent relations

Since the number m + 1 of terms in a recurrent relation (3) is at least 2, 2-term recurrencies are the simplest ones. Any such a relations is defined by two polynomials $P_0(y)$ and $P_1(y)$ of degree d + 1:

$$P_0(n)a_n = P_1(n+1)a_{n+1}.$$
(8)

Without loss of generality we again assume that the leading coefficient of $P_1(y)$ is 1.

Definition 4.4.1 Denote by

$$G_{d+1}(\alpha,\beta;w) = G_{d+1} \left(\begin{array}{c} \alpha_1, \dots, \alpha_{d+1} \\ \beta_1, \dots, \beta_{d+1} \end{array}; w\right)$$

the series

$$\sum_{n\geq 0} \frac{\prod_{i=1}^{d+1} \Gamma(\alpha_i)}{\prod_{i=1}^{d+1} \Gamma(\beta_i)} \times \left(\frac{\prod_{i=1}^{d+1} \Gamma(\beta_i+n)}{\prod_{i=1}^{d+1} \Gamma(\alpha_i+n)}\right) w^n.$$

which is the generalized hypergeometric function with parameters $\alpha_1, \ldots, \alpha_{d+1}, \beta_1, \ldots, \beta_{d+1}$. (This is a slight modification of the well-known generalized hypergeometric function $_{d+1}F_d$ (see [27, 33]).)

Proposition 4.4.2 Assume that

$$P_1(y) = \prod_{i=1}^{n} (y + \alpha_i);$$
$$P_0(y) = \lambda \prod_{i=1}^{n} (y + \beta_i)$$

then the function $G_{d+1}(\alpha,\beta;\lambda z)$ is a formal solution of the differential equation

$$\mathcal{P}\Phi = (P_1(\Theta) - zP_0(\Theta))\Phi = 0.$$

Consider now the case when \mathcal{D} is a *MU*-operator, i.e., $P_1(y) = y^{d+1}$, and the recurrent relation has the form

$$(n+1)^{d+1}a_{n+1} = P_0(n)a_n.$$

Then for the power series $\Psi(z) = \sum_{i \ge 1} b_i z^i$ which is the solution to

$$\mathcal{D}_{\Theta}'\Phi_0(z) + \mathcal{P}\Psi(z) = 0,$$

where

$$\Phi_0(z) = \sum_{i=0}^{\infty} a_i z^i,$$

is a regular solution to $\mathcal{P}\Psi = 0$, the coefficients $\{b_i\}$ satisfy the recurrent relation

$$n^{d+1}b_n = P_1(n-1)b_{n-1} + P_1'(n-1)a_{n-1} - (d+1)n^d a_n.$$

4.5 *d*-point Yukawa functions

Let $\pi : V_z \to S$ be a 1-parameter family of Calabi-Yau *d*-folds, where $S = \text{Spec } \mathbf{C}[[z]]$. Let *T* be the corresponding monodromy transformation acting on $H_d(W_z, \mathbf{C}), T_u$ the unipotent part part of *T*, $N = Log T_u$.

Definition 4.5.1 The family V_z is said to have the maximal unipotent monodromy at z = 0 if the weight filtration

$$0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{d-1} \subset W_d = H^d(V_z, \mathbf{C})$$

defined by N is orthogonal to the Hodge filtration $\{F^i\}$, i.e.,

$$H^d(V_z, \mathbf{C}) = W_i^{\perp} \oplus F^{d-i} \ i = 0, \dots, d.$$

(This is similar to definitions given in [17, 31].)

Assume that the family V_z/S has the maximal unipotent monodromy at z = 0 and dim $F^i/F^{i+1} = 1$ for i = 0, ..., d. Then the Jordan normal form of N has exactly one sell of size $(d + 1) \times (d + 1)$. This means that there exists a d-cycle $\gamma \in H_d(V_z, \mathbb{Z})$ such that $\gamma, N\gamma, ..., N^d\gamma$ are linearly independent in $H_d(V_z, \mathbb{Z})$, and $N^d\gamma = \gamma_0$ is a monodromy invariant d-cycle. Take a 1-parameter family $\omega(z)$ of holomorphic d-forms on W_z . It is well-known that the periods of $\omega(z)$ over the d-cycles in $H_d(V_z, \mathbb{C})$ satisfy a Picard-Fuchs differential equation of order d + 1 defined by some differential MU-operator

$$\mathcal{P} = \Theta^{d+1} + C_d(z)\Theta^d + \dots + C_0(z).$$
(9)

Definition 4.5.2 Define the coupling functions $W_{k,l}(z)$ $(k, l \ge 0, k, l \in \mathbf{Z})$ as follows

$$W_{k,l} = \int_{V_z} \Theta^k \omega(z) \wedge \Theta^l \omega(z)$$

(By definition, we put $\Theta^0 = 1$ to be the identity differential operator.)

Definition 4.5.3 [28] The coupling function $W_{d,0}$ is called *unnormalized d-point Yukawa* function.

Proposition 4.5.4 The coupling functions $W_{k,l}(z)$ satisfy the properties

(i) $W_{k,l}(z) = (-1)^d W_{l,k};$ (ii) $W_{k,l}(z) = 0$ for k + l < d;(iii) $\Theta W_{k,l}(z) = W_{k+1,l}(z) + W_{k,l+1}(z).$ (iv) $W_{d+k+1,0}(z) + C_d(z) W_{d+k,0}(z) + \dots + C_0(z) W_{k,0}(z) = 0.$

Proof. The statements follow immediately from the properties of the cup-product and from the Griffiths transversality property.

Theorem 4.5.5 The d-point Yukawa function $W_{d,0}(z)$ satisfies the linear differential equation of order one

$$\Theta W_{d,0}(z) + \frac{2}{d+1} C_d(z) W_{d,0} = 0.$$
(10)

Proof. By 4.5.4(ii), we have

$$W_{d-i,i}(z) + W_{d-i-1,i+1}(z) = 0$$
 for $i = 0, 1, \dots, d$. (11)

Therefore, $W_{d,0}(z) = (-1)^i W_{d-i,i}$. On the other hand, by 4.5.4(iii), we have

$$\Theta W_{d-i,i} = W_{d-i+1,i}(z) + W_{d-i,i+1}(z) \text{ for } i = 0, 1, \dots, d.$$
(12)

It follows from (11) and (12) that

$$k\Theta W_{d,0}(z) = \sum_{i=0}^{k-1} (-1)^i W_{d-i,i}(z) = W_{d+1,0}(z) + (-1)^{k-1} W_{d-k+1,k}(z).$$
(13)

CASE I: d is odd. Since

$$W_{\frac{d+1}{2},\frac{d+1}{2}}(z) = 0$$
 (4.5.4(i)).

we obtain

$$\Theta W_{\frac{d+1}{2},\frac{d-1}{2}}(z) = W_{\frac{d+3}{2},\frac{d-1}{2}}(z).$$

Using (11) and (13) for k = (d-1)/2, we obtain

$$\frac{(d+1)}{2}\Theta W_{d,0}(z) = W_{d+1,0}(z).$$

By 4.5.4(ii) and (iv), this implies the equation (10) for $W_{d,0}(z)$.

CASE I: d is odd. One has

$$\Theta W_{\frac{d}{2},\frac{d}{2}}(z) = W_{\frac{d+2}{2},\frac{d}{2}}(z) + W_{\frac{d}{2},\frac{d+2}{2}}(z) = 2W_{\frac{d+2}{2},\frac{d}{2}}(z).$$

Using (11) and (13) for k = d/2, we obtain

$$(d+1)\Theta W_{d,0}(z) = 2W_{d+1,0}(z).$$

The latter again implies the same linear differential equation for $W_{d,0}(z)$. Corollary 4.5.6

$$W_{d,0}(z) = c_0 \exp\left(-\frac{2}{d+1}\int_0^z C_d(v)\frac{dv}{v}\right)$$

for some nonzero constant $c_0 = W_{d,0}(0)$.

Example 4.5.7 Assume that $\mathcal{P} = \Theta^{d+1} - zP_0(\Theta)$ be the *MU*-operator corresponding to a 2-term recurrent relation $(n+1)^{d+1}a_{n+1} = P_0(n)a_n$, where $P_0(y) = \lambda y^{d+1} + \cdots$. Then the Yukawa *d*-point function $W_{d,0}(z)$ equals

$$W_{d,0}(z) = \frac{c_0}{1 - \lambda z},$$

i.e., $W_{d,0}(z)$ is a rational function in z.

4.6 Multidimensional Picard-Fuchs differential systems with a symmetry group

So far we considered only the case of the 1-parameter family of Calabi-Yau *d*-folds V_z such that dim $F^i/F^{i+1} = 1$ for i = 0, ..., d. It is easy to see that the same methods can be applied to the case dim $F^i/F^{i+1} \ge 1$ provided V_z has a large authomorphisms group.

Proposition 4.6.1 Let V_z be a 1-parameter family of Calabi-Yau d-folds with dim $F^i/F^{i+1} \ge 1$. Assume that there exists an action of a finite group G on V_z such that the G-invariant part $(F^i/F^{i+1})^G$ is 1-dimensional for all i = 0, ..., d. Then the holomorphic differential d-form $\omega(z)$ again satisfies the Picard-Fuchs differential equation of order d + 1.

Proof. The statement immediately follows from the fact that the cohomology classes of $\omega(z), \Theta\omega(z), \ldots, \Theta^d\omega(z)$ form the basis of the *G*-invariant subspace $H^d(V_z, \mathbf{C})^G \subset H^d(V_z, \mathbf{C})$.

5 Calabi-Yau complete intersections in \mathbf{P}^N

5.1 Rational curves and generalized hypergeometric series

Let V be a complete intersection of r-hypersurfaces V_1, \ldots, V_r of degrees d_1, \ldots, d_r in \mathbf{P}^{d+r} . Then V is a Calabi-Yau d-fold if and only if $d+r+1 = d_1 + \ldots + d_r$. A rational curve C of degree n in \mathbf{P}^{d+r} has nd_i intersection points with a generic hypersurface V_i . On the other hand, there exists a degeneration of every divisor V_i into the union of d_i hyperplanes. Each of these hyperplanes has n intersection points with C. This motivates the definition of the corresponding generalized hypergeometric series $\Phi_0(z)$ as

$$\sum_{i=0}^{\infty} \frac{(nd_1!)}{(n!)^{d_1}} \cdots \frac{(nd_r!)}{(n!)^{d_r}} z^n$$
(14)

The coefficients

$$a_n = \frac{(nd_1!)\cdots(nd_r!)}{(n!)^{d+r+1}}$$

satisfy the recurrent relation

$$(n+1)^{d+1}a_{n+1} = P(n)a_n$$

where P(y) is the polynomial of degree d + 1:

$$P(n) = \frac{(nd_1 + d_1)!}{(nd_1)!} \cdots \frac{(nd_r + d_r)!}{(nd_r)!} (n+1)^{-r} = \lambda n^{d+1} + \cdots$$

In particular, the leading coefficient of P(y) is $\lambda = \prod_{i=1}^{r} (d_i)^{d_i}$.

Example 5.1.1 Let V be a complete intersection of two cubics in \mathbf{P}^5 . The corresponding generalized hypergeometric series is

$$\Phi_0(z) = \sum_{n \ge 0} \frac{(3n!)^2}{(n!)^6} z^n.$$

This series was found in [25] using the explicit construction of mirrors for V by orbifolding the 1-parameter family of special complete intersections of two cubics in \mathbf{P}^5 :

$$Y_1^3 + Y_2^3 + Y_3^3 = 3\psi Y_4 Y_5 Y_6;$$

$$Y_4^3 + Y_5^3 + Y_6^3 = 3\psi Y_1 Y_2 Y_3,$$

by an abelian group G of order 81. where $z = (3\psi)^{-6}$.

We will give another interpretation of the construction of mirrors V' for V which immediately implies that $\Phi_0(z)$ is the monodromy invariant period for the regular differential 3-form on V'.

Let $Z_{f_1f_2}$ be the complete intersection of two hypersurfaces in a 5-dimensional algebraic torus $\mathbf{T} = \operatorname{Spec}[X_1^{\pm 1}, \ldots, X_5^{\pm 1}]$ defined by the Laurent polynomials

$$f_1(u, X) = 1 - (u_1X_1 + u_2X_2 + u_3X_3), \ f_2(u, X) = 1 - (u_4X_4 + u_5X_5 + u_6(X_1 \cdots X_5)^{-1}).$$

We define the differential 3-form ω on $Z_{f_1f_2}$ as the residue of the rational differential 5-form on **T**:

$$\omega = \frac{1}{(2\pi\sqrt{-1})} \operatorname{Res} \frac{1}{f_1(u,X) f_2(u,X)} \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_5}{X_5}.$$

Let $z = u_1 \cdots u_6$. By residue theorem, we obtain

$$\Phi_0(z) = \frac{1}{(2\pi\sqrt{-1})} \int_{|X_i|=1} \frac{1}{f_1(u,X)f_2(u,X)} \frac{dX_1}{X_1} \wedge \dots \wedge \frac{dX_5}{X_5}$$

In this interpretation, the mirrors for V are smooth Calabi-Yau compactifications of of affine 3-folds $Z_{f_1f_2}$.

The equivalence between the above two construction of mirrors for V follows by the substitution

$$X_1 = Y_1^3 / (Y_3 Y_4 Y_5), \ X_2 = Y_2^3 / (Y_3 Y_4 Y_5), \ X_3 = Y_1^3 / (Y_3 Y_4 Y_5),$$
$$X_4 = Y_4^3 / (Y_1 Y_2 Y_3), \ X_5 = Y_5^3 / (Y_1 Y_2 Y_3),$$
$$u_1 = \dots = u_6 = (3\psi)^{-1}.$$

Proposition 5.1.2 The normalized Yukawa d-differential for Calabi-Yau complete intersections has the form

$$\mathcal{W}_d = \frac{d_1 \cdots d_r}{(1 - \lambda z) \Phi_0^2(z)} \cdot (\frac{dz}{z})^{\otimes d},$$

where d_1, \ldots, d_i are degrees of hypersurfaces.

Proof. The statement follows from 4.5.7 and the normalizing condition $d_1 \cdots d_r = W_d(0)$.

5.2 The construction of mirrors

Let V be a d-dimensional Calabi-Yau complete intersection of r hypersurfaces of degrees d_1, \ldots, d_r in \mathbf{P}^{d+r} . We propose the explicit construction of candidats for mirrors with respect to V as follows:

Let $E = \{v_1, \ldots, v_{d+r+1}\}$ be a generating set in the (d+r)-dimensional lattice $N \cong \mathbb{Z}^{d+r}$ such that there exist the relation

$$v_1 + \dots + v_{d+r+1} = 0.$$

We divide E into a disjoint union of r subsets $E_i \subset E$ such that $\operatorname{Card} E_i = d_i$. For $i = 1, \ldots, r$, we define the Laurent polynomial $P_i(u, X)$ in variables X_1, \ldots, X_{d+r} as

$$P_i(X) = 1 - (\sum_{v_j \in E_i} u_j X^{v_j}),$$

where u_1, \ldots, u_{d+r+1} are independent parameters. We denote by V' a Calabi-Yau compactification of d-dimensional affine complete intersections Z in $\mathbf{T} = \operatorname{Spec} [X_1^{\pm 1}, \ldots, X_{d+r}^{\pm 1}]$

defined by the polynomials $P_1(u, X), \ldots, P_r(u, X)$ with sufficiently general coefficients u_i . It is easy to see that up to an isomorphism the affine variaties $Z \subset \mathbf{T}$ depend only on $z = u_1 \cdots u_{d+r+1}$. Thus, we have obtained a 1-parameter family of d-dimensional varieties V'.

Conjecture 5.2.1 The 1-parameter family of d-dimensional varieties V' yields the mirror family for V.

This conjecture is motivated by the combinatorial interpretation proposed in [1] of the well-known construction of mirrors for hypersurfaces of degree d + 2 in \mathbf{P}^{d+1} (see [16]). On the other hand, the conjecture is supported by the following property:

Proposition 5.2.2 The hypergeometric series $\Phi_0(z)$ in (14) is the monodromy invariant period function of the holomorphic d-form ω on V'.

Proof. The statement follows from the equality

$$\sum_{i=0}^{\infty} \frac{(nd_1!)}{(n!)^{d_1}} \cdots \frac{(nd_r!)}{(n!)^{d_r}} z^n = \frac{1}{(2\pi\sqrt{-1})^{d+r}} \int_{|X_j|=1} \frac{1}{P_1(X)\cdots P_r(X)} \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_{d+r}}{X_{d+r}}.$$

6 Complete intersections in toric varieties

6.1 The generalized hypergeometric series Φ_0

Let N be a free abelian group of rank (d+r). Consider r finite sets

$$E_i = \{v_{i,1}, \dots, v_{i,k_i}\}, \ i = 1, \dots, r$$

consisting of elements $v_{i,j} \in N$. Let E be the union $E_1 \cup \cdots \cup E_r$.

We put $k = \text{Card } E = k_1 + \ldots + k_r$ and assume that E generates the group N. Let R(E) be the subgroup in \mathbb{Z}^n consisting of all integral vectors $\lambda = \{\lambda_{i,j}\}$ such that

$$\sum_{i=1}^r \sum_{j=1}^{k_i} \lambda_{i,j} v_{i,j} = 0.$$

We denote by $R^+(E)$ submonoid in R(E) consisting of all $\lambda = \{\lambda_{i,j}\} \in R(E)$ such that $\lambda_{i,j} \geq 0$.

Definition 6.1.1 Let $u_{i,j}$ be k independent complex variables parametrized by k integral vectors $v_{i,j}$. Define the power series $\Phi_0(u)$ as

$$\Phi_0(u) = \sum_{\lambda \in R^+(E)} \prod_{i=1}^r (\sum_{j=1}^{k_i} \lambda_{i,j})! \left(\prod_{j=1}^{k_i} \frac{u_{i,j}^{\lambda_{i,j}}}{(\lambda_{i,j})!} \right).$$

Let $\lambda^{(1)}, \ldots, \lambda^{(t)}$ be a **Z**-basis of the lattice R(E) such that every element $\lambda \in R^+(E)$ is a non-negative integral linear combination of $\lambda^{(i)}$. We define new r complex variables z_1, \ldots, z_s as follows

$$z_s = \prod_{i=1}^t \prod_{j=1}^{k_i} u_{i,j}^{\lambda_{i,j}^{(s)}}; \ s = 1, \dots, t.$$

Thus, the series $\Phi_0(u)$ can be rewritten as the power series $\Phi_0(z)$ in t variables z_1, \ldots, z_t .

Example 6.1.2 Let $E = \{v_1, \ldots, v_{d+1}\}$ be vectors generating *d*-dimensional lattice *N* and satisfying the integral relation $v_1 + \cdots + v_{d+1} = 0$, i.e., the group R(E) is generated by the vector $(1, \ldots, 1) \in \mathbb{Z}^{d+1}$. Then the corresponding generalized hypergeometric series is

$$\Phi_0(u) = \sum_{n \ge 0} \frac{(nd+n)!}{(n!)^{d+1}} (u_1 \cdots u_{d+1})^n = \sum_{n \ge 0} \frac{(nd+n)!}{(n!)^{d+1}} z^n = \Phi_0(z),$$

where $z = u_1 \cdots u_{d+1}$. The integral representation of this series is the monodromy invariant period function for mirrors of hypersurfaces of degree (d+1) in \mathbf{P}^d .

Definition 6.1.3 Let **T** be a (d + r)-dimensional algebraic torus with the Laurent coordinates $X = (X_1, \ldots, X_{d+r})$. We define r Laurent polynomials $P_{E_1}(X), \ldots, P_{E_r}(X)$ as follows

$$P_{E_i}(X) = 1 - \sum_{v_{i,j} \in E_i} u_{i,j} X^{v_{i,j}}.$$

Proposition 6.1.4 The series $\Phi_0(u)$ admits the following integral representation

$$\Phi_0(u) = \frac{1}{(2\pi\sqrt{-1})^{d+r}} \int_{|X_j|=1} \frac{1}{P_{E_1}(X)\cdots P_{E_r}(X)} \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_{d+r}}{X_{d+r}}.$$

Proof. The statement follows immediately from the residue formula.

6.2 Calabi-Yau complete intersections

Let \mathbf{P}_{Σ} be a quasi-smooth (d + r)-dimensional projective toric variety defined by a (d + r)-dimensional simplicial fan Σ . Assume that there exist r line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ such that each \mathcal{L}_i is generated by global sections and the tensor product

$$\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r$$

is isomorphic to the anticanonical bundle on \mathcal{K}^{-1} on \mathbf{P}_{Σ} . If V_i is the set of zeros of a generic global section of \mathcal{L}_i , then the complete intersection $V = V_1 \cap \cdots \cap V_r$ is a *d*-dimensional Calabi-Yau variety having only Gorenstein toroidal singularities which are analytically isomorphic to toric singularities of \mathbf{P}_{Σ} .

Now let $E = \{v_1, \ldots, v_k\}$ be the set of all generators of 1-dimensional cones in Σ . Denote by D_j the toric divisor on \mathbf{P}_{Σ} corresponding to v_j . Notice that

$$\mathcal{K}^{-1} = \bigotimes_{j=1}^k \mathcal{O}_{\mathbf{P}_{\Sigma}}(D_j).$$

Following a suggestion of Yu. I. Manin, we assume that one can represent E as a disjoint union

$$E = E_1 \cup \cdots \cup E_r$$

such that the line bundle \mathcal{L}_i is isomorphic to the tensor product

$$\otimes_{v_j \in E_i} \mathcal{O}_{\mathbf{P}_{\Sigma}}(D_j).$$

The elements of the group R(E) can be identified with the homology classes of 1dimensional algebraic cycles on \mathbf{P}_{Σ} . Moreover, one has the following property

Proposition 6.2.1 Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be an arbitrary element of R(E) representing the class of an algebraic 1-cycle C. Then

$$\lambda_i = \langle D_i, C \rangle, \ i = 1, \dots, k.$$

We can always choose a **Z**-basis $\lambda^{(1)}, \ldots, \lambda^{(t)}$ of R(E) such that every *effective* 1-cycle on \mathbf{P}_{Σ} is a non-negative linear combination of the elements $\lambda^{(1)}, \ldots, \lambda^{(t)}$. Since the submonoid $R^+(E)$ consists of classes of *nef-curves*, this implies that every element of $R^+(E)$ is also a non-negative linear combination of $\lambda^{(1)}, \ldots, \lambda^{(t)}$. This allows us to rewrite the series $\Phi_0(u)$ in t algebraically independent variables $z_1 \ldots, z_t$ $(t = \operatorname{rk} R(E))$.

Corollary 6.2.2 The series $\Phi_0(z)$ can be interpreted via the intersection numbers of classes [C] of curves C on \mathbf{P}_{Σ} as follows

$$\Phi_0(z) = \sum_{[C]\in R^+(E)} \frac{(\langle V_1, C \rangle)! \cdots (\langle V_r, C \rangle)!}{\langle D_1, C \rangle! \cdots \langle D_k, C \rangle!} z^{[C]},$$

where $z^{[C]} = z_1^{c_1} \dots z_t^{c_t}$, $[C] = c_1 \lambda^{(1)} + \dots + c_t \lambda^{(t)}$.

6.3 General conjectures

Let V be a d-dimensional Calabi-Yau complete intersection of hypersurfaces V_1, \ldots, V_r in a (d+r)-dimensional quasi-smooth toric variety defined by a simplicial fan Σ . Choose a **Z**-basis $\lambda^{(1)}, \ldots, \lambda^{(t)}$ in R(E) such that the classes of all effective 1-cycles have non-negative integral coordinates. We assume that the divisors V_1, \ldots, V_r are numerically effective (in particular, they are not assumed to be necessary ample divisors). We assume also that the following conditions are satisfied:

(i) V is smooth;

(ii) the restriction mapping $\operatorname{Pic} \mathbf{P}_{\Sigma} \to \operatorname{Pic} V$ is injective.

In this situation, there exist two flat A-model connections: the connection ∇_{AP} on $H^*(\mathbf{P}_{\Sigma})$ and the connection ∇_{AV} on $H^*(V, \mathbf{C})$. Let \tilde{H}^i be the image of $H^i(\mathbf{P}_{\Sigma}, \mathbf{C})$ in $H^i(V, \mathbf{C})$. The connection ∇_{AP} defines the quantum variation on cohomology of toric variety \mathbf{P}_{Σ} . It follows from the result in [3] the following.

Proposition 6.3.1 The complex coordinates z_1, \ldots, z_t on \tilde{H}^2 can be identified with flat coordinates with respect to ∇_{AP} .

Conjecture 6.3.2 The generalized hypergeometric series $\Phi_0(z)$ as a function of ∇_{AP} -flat z- coordinates on \tilde{H}^2 is a solution of the differential system \mathcal{D} defined by the A-model connection ∇_{AV} on \tilde{H}^2 which defines the quantum variation of Hodge structures on the subring in $\bigoplus_{i=0}^{d} H^{2i}(V, \mathbf{C})$ generated by restrictions of the classes in Pic \mathbf{P}_{Σ} to V.

Remark 6.3.3 One can check in many examples that the differential system \mathcal{D} is already defined by the generalized hypergeometric series $\Phi_0(z)$. Probably there exists a general explanation of this fact.

Conjecture 6.3.4 The ∇_{AV} -flat coordinates q_1, \ldots, q_t on \tilde{H}^2 are defined as

$$q_i = \exp(\Phi_i(z)/\Phi_0(z)), \ i = 1, \dots, t,$$

where $\Phi_i(z)$ is a logarithmic solution to the differential system \mathcal{D} having the form

$$\Phi_i(z) = (\log z_i)\Phi_0(z) + \Psi_i(z), \ \Psi_i(0) = 0$$

for some regular at z = 0 power series $\Psi_i(z)$.

Moreover, all coefficients of the expansion of ∇_{AV} -flat coordinates q_i as power series of ∇_{AP} -flat z-coordinates are integers.

Remark 6.3.5 This conjecture establishes a general method for normalizing the logarithmic solutions defining the canonical q-coordinates for the differential system \mathcal{D} . There are two motivations for this conjecture. First, the conjecture is true for all already known examples of q-coordinates for Picard-Fuchs equations corresponding to Calabi-Yau complete intersections in products of weighted projective spaces (see examples in the remaining part of the paper). Second, the Lefschetz theorem and the calculation of the quantum cohomolgy ring of toric varieties [3] imply the relation

$$q_i = z_i + O(|z|^2), \ (i = 1, \dots, t).$$

Conjecture 6.3.6 Assume that V has dimension 3. Let $K_{i,j,k}(z)$ be structure constants defining the A-model ∇_{AV} connection in the z-coordinates. Then

$$\Phi_0^2(z)K_{i,j,k}(z)$$

is a rational function in z-coordinates.

Conjecture 6.3.7 The mirror Calabi-Yau varieties with respect to V are Calabi-Yau compactifications of the complete intersection of the affine hypersurfaces in the (d+r)-dimension nal algebraic torus **T** defined by the equations

$$P_{E_1}(X) = \dots = P_{E_r}(X) = 0.$$

Remark 6.3.8 Recall that two Calabi-Yau *d*-folds V and V' are called mirror symmetric if $h^{p,d-p}(V) = h^{d-p,d-p}(V')$ and the superconformal field theories corresponding to V and V' are isomorphic. It was proposed in [1] a general method for constructing pairs of mirror symmetric Calabi-Yau hypersurfaces in toric varieties based on the duality among so called *reflexive polyhedra* Δ and Δ^* . However, the equality $h^{1,1}(\hat{Z}_f) = h^{d-1,1}(\hat{Z}_g)$ for the pair of Calabi-Yau *d*-folds \hat{Z}_f and \hat{Z}_g corresponding to the polyhedra Δ and Δ^* are not sufficient to prove the mirror duality between \hat{Z}_f and \hat{Z}_g in full strength. One needs to prove more: the isomorphism between the quantum cohomology of \hat{Z}_f and \hat{Z}_g . Since the quantum cohomology are defined by the canonical form of the A-model connection ∇_A in *q*-coordinates, Conjecture 6.3.2 and Proposition 6.1.4 yield more evidence for validity of Conjecture 6.3.7. We give below one example showing that Conjecture 6.3.7 agrees with an orbifold construction of mirrors for complete intersections in product of projective spaces inspired by superconformal field theories.

Example 6.3.9 Let V be a Calabi-Yau complete intersection of two hypersurfaces of degrees (3,0) and (1,3) in the product $\mathbf{P}^3 \times \mathbf{P}^2$.

It is known that the mirrors for V can be obtained by orbifolding the complete intersection of two special hypersurfaces

$$S_1T_1^3 + S_2T_2^3 + S_3T_3^3 = \phi S_4T_1T_2T_3;$$
$$S_1^3 + S_2^3 + S_3^3 + S_4^3 = \psi S_1S_2S_3$$

by the group G of order 27, where $(S_1 : S_2 : S_3 : S_4)$ and $(T_1 : T_2 : T_3)$ are the homogeneous coordinates on \mathbf{P}^3 and \mathbf{P}^2 respectively.

On the other hand, the 5-dimensional fan Σ defining $\mathbf{P}^3 \times \mathbf{P}^2$ has 7 generators $\{v_1, \ldots, v_7\} = E$ satisfying the relations

$$v_1 + v_2 + v_3 + v_6 = v_4 + v_5 + v_7 = 0.$$

We choose vectors v_1, \ldots, v_5 as the basis of the 5-dimensional lattice N. The complete intersection V is defined by dividing E into two subsets $E_1 = \{v_1, v_2, v_3\}$ and $E_2 = \{v_4, v_5, v_6, v_7\}$. The corresponding polynomials $P_{E_1}(X)$ and $P_{E_2}(X)$ are

$$P_{E_1} = 1 - (u_1 X_1 + u_2 X_2 + u_3 X_3),$$

$$P_{E_2} = 1 - (u_4 X_4 + u_5 X_5 + u_6 (X_1 X_2 X_3)^{-1} + u_7 (X_4 X_5)^{-1}).$$

We obtain the equivalence between two construction of mirrors by putting $u_1 = u_2 = u_3 = \phi^{-1}$, $u_4 = u_5 = u_6 = u_7 = \psi^{-1}$, and

$$X_{1} = \frac{S_{1}T_{1}^{2}}{S_{4}T_{2}T_{3}}, X_{2} = \frac{S_{2}T_{2}^{2}}{S_{4}T_{1}T_{3}}, X_{3} = \frac{S_{3}T_{3}^{2}}{S_{4}T_{1}T_{2}},$$
$$X_{4} = \frac{S_{1}^{2}}{S_{2}S_{3}}, X_{5} = \frac{S_{2}^{2}}{S_{1}S_{3}}.$$

6.4 Calabi-Yau 3-folds with $h^{1,1} = 1$

We consider below examples of the generalized hypergeometric series corresponding to smooth Calabi-Yau complete intersections V of r hypersurfaces in a toric variety \mathbf{P}_{Σ} such that $h^{1,1}(V) = 1$. By Lefschetz theorem, $h^{1,1}(\mathbf{P}_{\Sigma})$ must be also 1. So Σ is a (r + 3)dimensional fan with (r + 4) generators. There exists the unique primitive integral linear relation $\Sigma \lambda_i v_i = 0$ among the generators $\{v_i\}$ of Σ , i.e., $\operatorname{rk} R(E) = 1$ where $E = \{v_i\}$ is the whole set of generators of Σ (Card E = r + 4).

In all these examples the MU-operator \mathcal{P} has form

$$\mathcal{P} = \Theta^4 - \mu z (\Theta + \alpha_1) (\Theta + \alpha_2) (\Theta + \alpha_3) (\Theta + \alpha_4),$$

where the numbers $\alpha_1, \ldots, \alpha_4$ are positive rationals satisfying the relations

$$\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3 = 1.$$

The Yukawa 3-differential in z-coordinate has form

$$\mathcal{W}_3 = \frac{W(0)}{(1-\mu z)\Phi_0^2(z)} \left(\frac{dz}{z}\right)^{\otimes 3}$$

Example 6.4.1 Hypersurfaces in weighted projective spaces: In this case we obtain Calabi-Yau hypersurfaces in the following weighted projective spaces $\mathbf{P}(\lambda_1, \dots, \lambda_5)$

$(\lambda_1,\ldots,\lambda_5)$	$\Phi_0(z)$	W(0)	μ	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$
(1, 1, 1, 1, 1)	$\sum_{n\geq 0} \frac{(5n)!}{(n!)^5} z^n$	5	5^{5}	(1/5, 2/5, 3/5, 4/5)
(2, 1, 1, 1, 1)	$\sum_{n \ge 0} \frac{(6n)!}{(n!)^4 (2n!)} z^n$	3	$2^{5}3^{6}$	(1/6, 2/6, 4/6, 5/6)
(4, 1, 1, 1, 1)	$\sum_{n \ge 0} \frac{(8n)!}{(n!)^4 (4n!)} z^n$	2	2^{18}	(1/8, 3/8, 5/8, 7/8)
(5, 2, 1, 1, 1)	$\sum_{n \ge 0} \frac{(10n)!}{(n!)^3 (2n!) (5n!)} z^n$	1	$2^{9}5^{6}$	(1/10, 3/10, 7/10, 9/10)

The q-expansion of the Yukawa 3-point function and predictions n_d for number of rational curves on these hypersurfaces were obtained in [29, 20, 13].

Example 6.4.2 Complete intersections in ordinary projective spaces: Let V_{d_1,\ldots,d_r} denotes the complete intersection of hypersurfaces of degrees d_1,\ldots,d_r .

	$\Phi_0(z)$	W(0)	μ	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$
$V_{3,3} \subset \mathbf{P}^5$	$\sum_{n \ge 0} \frac{((3n)!)^2}{(n!)^6} z^n$	9	3^{6}	(1/3, 1/3, 2/3, 2/3)
$V_{2,4} \subset \mathbf{P}^5$	$\sum_{n \ge 0} \frac{(2n)!(4n)!}{(n!)^6} z^n$	8	2^{10}	(1/4, 2/4, 2/4, 3/4)
$V_{2,2,3} \subset \mathbf{P}^6$	$\sum_{n \ge 0} \frac{((2n)!)^2 (3n)!}{(n!)^7} z^n$	12	$2^4 3^3$	(1/3, 1/2, 1/2, 2/3)
$V_{2,2,2,2} \subset \mathbf{P}^8$	$\sum_{n \ge 0} \frac{((2n)!)^4}{(n!)^8} z^n$	16	2^{8}	(1/4, 1/4, 1/4, 1/4)

These Calabi-Yau complete intersections in ordinary projective spaces were considered by Libgober and Teitelbaum [25].

Example 6.4.3 Complete intersections in weighted projective spaces:

	$\Phi_0(z)$	W(0)	μ	$(lpha_1, lpha_2, lpha_3, lpha_4)$
$V_{4,4} \in \mathbf{P}(1,1,1,1,2,2)$	$\sum_{n=0}^{\infty} \frac{(4n!)^2}{(n!)^4 (2n!)^2} z^n$	4	2^{12}	(1/4, 1/4, 3/4, 3/4)
$V_{6,6} \in \mathbf{P}(1, 1, 2, 2, 3, 3)$	$\sum_{n=0}^{\infty} \frac{(6n!)^2}{(n!)^2 (2n!)^2 (3n!)^2} z^n$	1	$2^{8}3^{6}$	(1/6, 1/6, 5/6, 5/6)
$V_{3,4} \in \mathbf{P}(1,1,1,1,1,2)$	$\sum_{n=0}^{\infty} \frac{(4n!)(3n!)}{(n!)^5(2n!)} z^n$	6	$2^{6}3^{3}$	(1/4, 1/3, 2/3, 3/4)
$V_{2,6} \in \mathbf{P}(1, 1, 1, 1, 1, 3)$	$\sum_{n=0}^{\infty} \frac{(6n!)(2n!)}{(n!)^5(3n!)} z^n$	4	$2^{8}3^{3}$	(1/6, 1/2, 1/2, 5/6)
$V_{4,6} \in \mathbf{P}(1, 1, 1, 2, 2, 3)$	$\sum_{n=0}^{\infty} \frac{(6n!)(4n!)}{(n!)^3 (2n!)^2 (3n!)} z^n$	2	$2^{10}3^3$	(1/6, 1/4, 3/4, 5/6)

The coefficients of the Yukawa 3-point function $K_q^{(3)}$ for these five examples of Calabi-Yau 3-folds V having the Hodge number $h^{1,1}(V) = 1$ were obtained by A. Klemm and S. Theisen [21].

7 Calabi-Yau 3-folds in $\mathbf{P}^2 \times \mathbf{P}^2$

7.1 The generalized hypergeometric series Φ_0

Calabi-Yau 3-folds V in $\mathbf{P}^2 \times \mathbf{P}^2$ are hypersurfaces of degree (3,3). The homology classes of rational curves on $\mathbf{P}^2 \times \mathbf{P}^2$ are parametrized by pairs of integers (l_1, l_2) . Let γ_1, γ_2 be the homology classes of (1, 0)-curves and (0, 1)-curves respectively. Then for any Kähler class η we put

$$z_i = \exp(-\int_{\gamma_i} \eta), \ (i = 1, 2).$$

The generalized hypergeometric series corresponding to the fan Σ defining $\mathbf{P}^2 \times \mathbf{P}^2$ is

$$\Phi_0(z_1, z_2) = \sum_{l_1, l_2 \ge 0} \frac{(3l_1 + 3l_2)!}{(l_1!)^3 (l_2!)^3} z_1^{l_1} z_2^{l_2}.$$

There are obvious two recurrent relations for the coefficients a_{l_1,l_2} of the series

$$\Phi_0(z_1, z_2) = \sum_{l_1, l_2 \ge 0} a_{l_1, l_2} z_1^{l_1} z_2^{l_2} :$$

$$(l_1+1)^3 a_{l_1+1,l_2} = (3l_1+3l_2+1)(3l_1+3l_2+2)(3l_1+3l_2+3)a_{l_1,l_2};$$

$$(l_2+1)^3 a_{l_1,l_2+1} = (3l_1+3l_2+1)(3l_1+3l_2+2)(3l_1+3l_2+3)a_{l_1,l_2};$$

Let

$$\Theta_1 = z_1 \frac{\partial}{\partial z_1}, \ \Theta_2 = z_2 \frac{\partial}{\partial z_2}$$

Then the function $\Phi_0(z_1, z_2)$ satisfies the Picard-Fuchs differential system \mathcal{D} :

$$\left(\Theta_1^3 - z_1(3\Theta_1 + 3\Theta_2 + 1)(3\Theta_1 + 3\Theta_2 + 2)(3\Theta_1 + 3\Theta_2 + 3)\right)\Phi_0 = 0;$$

$$\left(\Theta_2^3 - z_2(3\Theta_1 + 3\Theta_2 + 1)(3\Theta_1 + 3\Theta_2 + 2)(3\Theta_1 + 3\Theta_2 + 3)\right)\Phi_0 = 0.$$

The differential system \mathcal{D} has the maximal unipotent monodromy at $(z_1, z_2) = (0, 0)$. There are two uniquely determined regular at (0,0) functions $\Psi_1(z_1, z_2)$ and $\Psi_2(z_1, z_2)$ such that

$$(\log z_1)\Phi_0(z_1, z_2) + \Psi_1(z_1, z_2),(\log z_2)\Phi_0(z_1, z_2) + \Psi_2(z_1, z_2)$$

are solutions to \mathcal{D} , and $\Psi_1(0,0) = \Psi_2(0,0) = 0$. If we put

$$\Psi_j(z_1, z_2) = \sum_{\substack{l_1, l_2 \ge 0 \\ (l_1, l_2) \neq (0, 0)}} b_{l_1, l_2}^{(j)} z_1^{l_1} z_2^{l_2},$$

then one finds the coefficients $b_{l_1,l_2}^{(j)}$ from the simple recurrent relations based on 4.3.2. The q-coordinates q_1 , q_2 defined by the formulas

$$q_1 = z_1 \exp(\Psi_1/\Phi_0), \ q_2 = z_2 \exp(\Psi_2/\Phi_0)$$

are the power series with integral coefficients in z_1, z_2 of the form

$$q_j(z_1, z_2) = z_j \left(1 + \sum_{\substack{l_1, l_2 \ge 0 \\ (l_1, l_2) \neq (0, 0)}} c_{l_1, l_2}^{(j)} z_1^{l_1} z_2^{l_2} \right), \ j = 1, 2.$$

By symmetry, one has $c_{l_1,l_2}^{(1)} = c_{l_2,l_1}^{(2)}$.

7.2 Mirrors and the discriminant

Let f be the Laurent polynomial

$$f(X,u) = 1 - u_1 X_1 - u_2 X_2 - u_3 (X_1 X_2)^{-1} - u_4 X_3 - u_5 X_4 - u_6 (X_3 X_4)^{-1}.$$

Let γ_0 be a generator of $H_4((\mathbf{C}^*)^4, \mathbf{Z})$, i.e., the cycle defined by the condition $|X_i| = 1$ for $i = 1, \ldots 4$.

By the residue theorem, the integral

$$I(u) = \frac{1}{(2\pi\sqrt{-1})^4} \int_{\gamma_0} \frac{1}{f(X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}$$

is the power series

$$I(u) = \sum_{k,m \ge 0} \frac{(3k+3m)!}{(k!)^3 (m!)^3} (u_1 u_2 u_3)^k (u_4 u_5 u_6)^m.$$

Thus, putting $z_1 = u_1 u_2 u_3$; $z_2 = u_4 u_5 u_6$, we obtain exactly the generalized hypergeometric function $\Phi_0(z_1, z_2)$.

It was proved in [1] and [2] that the function I(u) can be considered as the monodromy invariant period of the holomorphic differential 3-form

$$\omega = \frac{1}{(2\pi\sqrt{-1})^4} \operatorname{Res} \frac{1}{f(X)} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \wedge \frac{dX_3}{X_3} \wedge \frac{dX_4}{X_4}.$$

for the family of Calabi-Yau 3-folds \hat{Z}_f which are smooth compactifications of the affine hypersurfaces Z_f in $(\mathbf{C}^*)^4$ defined by Laurent polynomial f. One has $h^{1,1}(\hat{Z}_f) = 83$, $h^{2,1}(\hat{Z}_f) = 2$. The coordinates z_1, z_2 are natural coordinates on the moduli space of Calabi-Yau 3-folds \hat{Z}_f .

The mirror construction helps to understand the the discriminant of the differential system \mathcal{D} as a polynomial function in z_1, z_2 .

By definition [15], the zeros of the discriminat are exactly those values of the coefficients $\{u_i\}$ of f(X) such that the system

$$f(X) = X_1 \frac{\partial}{\partial X_1} f(X) = X_2 \frac{\partial}{\partial X_2} (X) = X_3 \frac{\partial}{\partial X_3} f(X) = X_4 \frac{\partial}{\partial X_4} f(X) = 0$$

has a solution in the toric variety \mathbf{P}_{Δ} , where Δ is the Newton polyhedron of f. Since \mathbf{P}_{Δ} is isomorphic to the subvariety of \mathbf{P}^{6} defined as

$$\mathbf{P}_{\Delta} = \{ (Y_0 : \ldots : Y_6) \in \mathbf{P}^6 \mid Y_0^3 = Y_1 Y_2 Y_3, Y_0^3 = Y_4 Y_5 Y_6 \},\$$

or equivalently, the system of the homogeneous equations

$$u_0Y_0 + \dots + u_6Y_6 = u_1Y_1 - u_3Y_3 = u_2Y_2 - u_3Y_3 = u_4Y_4 - u_6Y_6 = u_5Y_5 - u_6Y_6 = 0;$$

 $Y_0^3 = Y_1Y_2Y_3 = Y_4Y_5Y_6$

has a non-zero solution.

If we put

$$A = u_3 Y_3, \ B = u_6 Y_6, \ C = u_0 Y_0$$

then the last system can be rewritten as

$$3A + 3B + C = A^3 + z_1 C^3 = B^3 + z_2 C^3 = 0.$$

So the discriminant the two-parameter family is the resultant of two binary homogeneous cubic equations in A and B:

$$27z_1(A+B)^3 - A^3 = 0, \ 27z_2(A+B)^3 - B^3 = 0.$$

Put $27z_1 = x$, $27z_2 = y$.

Proposition 7.2.1 The discriminant of the 2-parameter family of Calabi-Yau 3-folds \hat{Z}_f is

Disc
$$f = 1 - (x + y) + 3(x^2 - 7xy + y^2) - (x^3 + 3x^2y + 3xy^2 + y^3).$$

7.3 The diagonal one-parameter subfamily

We consider the diagonal one-parameter subfamily of Kähler structures η on V which are invariant under the natural involution on $H^{1,1}(V)$, i.e., we assume that

$$\int_{\gamma_1} \eta = \int_{\gamma_2} \eta$$

This is equivalent to the substitution $z = z_1 = z_2$.

Remark 7.3.1 In this case we obtain the one-parameter family of mirrors

$$f_{\psi}(X) = X_1 + X_2 + (X_1 X_2)^{-1} + X_3 + X_4 + (X_3 X_4)^{-1} - 3\psi = 0, \ \psi^3 = (27z)^{-1}$$

which is an analog to mirrors of quintic 3-folds [8].

It is easy to check that the discriminant of $f_{\psi}(X)$ vanishes exactly when $\psi = \alpha + \beta$, where $\alpha^3 = \beta^3 = 1$, i.e., $\psi^3 \in \{8, -1\}$, or $z \in \{-(3)^{-3}, (2 \cdot 3)^{-3}\}$.

The monodromy invariant period function is

$$F_0(z) = \Phi_0(z, z) = \sum_{n \ge 0} \left(\sum_{k+m=n} \frac{(3n)!}{(k!)^3 (m!)^3} \right) z^n.$$

It satisfies an ordinary Picard-Fuchs differential equation

$$\mathcal{D}$$
: $(\Theta^4 + \sum_{i=0}^3 C_i(z)\Theta^i)F(z) = 0, \ \Theta = z\frac{\partial}{\partial z}$

We compute the Picard-Fuch differential equation \mathcal{E} for $F_0(z)$ is from the recurrent formula for the coefficients

$$a_n = \sum_{k+m=n} \frac{(3n)!}{(k!)^3 (m!)^3} = \frac{(3n)!}{(n!)^3} (\sum_{k=0}^n \binom{n}{k}^3)$$

in the power expansion

$$F_0(z) = \sum_{n \ge 0} a_n z^n.$$

Proposition 7.3.2 ([34]) Let

$$b_n = \sum_{k=0}^n \binom{n}{k}^3.$$

Then the numbers b_n satisfy the recurrent relation

$$(n+1)^2 b_{n+1} = (7n^2 + 7n + 2)b_n + 8n^2 b_{n-1}.$$

Corollary 7.3.3 The numbers a_n satisfy the recurrent relation

 $(n+1)^4 a_{n+1} = 3(7n^2 + 7n + 2)(3n+2)(3n+1)a_n + 72(3n+2)(3n+1)(3n-1)(3n-2)a_{n-1}.$

Corollary 7.3.4 The monodromy invariant period function $F_0(y)$ is annihilated by the differential operator \mathcal{P} :

$$\Theta^4 - 3z(7\Theta^2 + 7\Theta + 2)(3\Theta + 1)(3\Theta + 2) - 72z^2(3\Theta + 5)(3\Theta + 4)(3\Theta + 2)(3\Theta + 1).$$

The last operator can be rewritten also as

$$(1-216z)(1+27z)\Theta^4 - 54z(7+432z)\Theta^3 - 3z(10584z+95)\Theta^2 - 48z(351z+2)\Theta - 12z - 2880z^2.$$

In particular, one has the coefficient

$$C_3(z) = \frac{-54z(7+432z)}{(1-216z)(1+27z)}.$$

The z-normalized Yukawa coupling $K_z^{(3)}$ is the solution to the differential equation

$$\frac{dK_z^{(3)}}{dz} = \frac{27(7+432z)}{(1-216z)(1+27z)}K_z^{(3)}.$$

Let *H* be the cohomology class in $H^2(V, \mathbb{Z})$ such that $\langle H, \gamma_1 \rangle = \langle H, \gamma_2 \rangle = 1$. Since $H^3 = 18$, we obtain the normalization condition

$$K_z^{(3)}(0) = 18$$

Applying the general algorithm in 4.3.3, we find the *q*-expansion of the *z*-coordinate

$$z(q) = q - 48q^2 - 18q^3 + 7976q^4 - 1697115q^5 + O(q^6),$$

and the q-expansion of the q-normalized Yukawa coupling is

$$K_q^{(3)} = 18 + 378q + 69498q^2 + 7724862q^3 + 1030043898q^4 + 132082090128q^5 + O(q^6).$$

We expect that

$$K_q^{(3)} = 18 + \sum_{d=1}^{\infty} \frac{n_d d^3 q^d}{1 - q^d}.$$

where n_d are predictions for numbers rational curves of degree d relative to the ample divisor of type (1, 1) on V. In particular, one has $n_1 = 378$.

7.4 Lines on a generic Calabi-Yau 3-fold in $P^2 \times P^2$

We show how to check the prediction for the number of lines on a generic Calabi-Yau 3-fold in $\mathbf{P}^2 \times \mathbf{P}^2$.

First we formulate one lemma which will be useful in the sequel.

Lemma 7.4.1 Let M be a complete algebraic variety, \mathcal{L}_1 and \mathcal{L}_2 two invertible sheaves on M such that the projectivizations $\mathbf{P}(\mathcal{L}_i) = \mathbf{P}(H^0(M, \mathcal{L}_i))$ (i = 1, 2) are nonempty. Define the morphism

$$p_{\lambda} : \mathbf{P}(\mathcal{L}_1) \times \mathbf{P}(\mathcal{L}_2) \to \mathbf{P}(\mathcal{L}_1 \otimes \mathcal{L}_2) = \mathbf{P}(H^0(M, \mathcal{L}_1 \otimes \mathcal{L}_2))$$

by the natural mapping

$$\lambda : H^0(M, \mathcal{L}_1) \otimes H^0(M, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2).$$

Then the pullback $p_{\lambda}^* \mathcal{O}(1)$ of the ample generator $\mathcal{O}(1)$ of the Picard group of $\mathbf{P}(\mathcal{L}_1 \otimes \mathcal{L}_2)$ is isomorphic to $\mathcal{O}(1,1)$ on $\mathbf{P}(\mathcal{L}_1) \times \mathbf{P}(\mathcal{L}_2)$.

Proof. The statement follows immediately from the fact that λ is bilinear. \Box

Proposition 7.4.2 A generic Calabi-Yau hypersurface in $\mathbf{P}^2 \times \mathbf{P}^2$ contains 378 lines relative to the $\mathcal{O}(1,1)$ -polarization.

Proof. There are two possibilities for the type of lines: (1, 0) and (0, 1). By symmetry, it is sufficient to consider only (1, 0)-lines whose projections on the second factor in $\mathbf{P}^2 \times \mathbf{P}^2$ are points. Let

$$\pi_2 : V \to \mathbf{P}^2$$

be the projection of V on the second factor. Then for every point $p \in \mathbf{P}^2$ the fiber $\pi_2^{-1}(p)$ is a cubic in $\mathbf{P}^2 \times p$. We want to calculate the number of those fibers $\pi_2^{-1}(p)$ which are unions of a line L and a conic Q in $\mathbf{P}^2 \times p$. The space of the reducible cubics $L \cup Q$ is isomorphic to the image $\Lambda \subset \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3))$ of the morphism

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(1)) \times \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2))) = \mathbf{P}^2 \times \mathbf{P}^5 \to \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3)).$$

By 7.4.1, Λ has codimension 2 and degree 21.

On the other hand, a generic Calabi-Yau hypersurface V defines a generic Veronese embedding

$$\phi : \mathbf{P}^2 \hookrightarrow \mathbf{P}^9 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(3)), \ \phi(p) = \pi_2^{-1}(p).$$

The degree of the image $\phi(\mathbf{P}^2)$ is 9. The number of (1, 0)-lines is the intersection number of two subvarieties $\phi(\mathbf{P}^2)$ and Λ in \mathbf{P}^9 , i.e., $9 \times 21 = 189$. Thus, the total amount of lines is $2 \times 189 = 378$.

8 Further examples

In this section we consider more examples of Calabi-Yau 3-folds V obtained as complete intersections in product of projective spaces. In all these examples for simplicity we restrict ourselves to one-parameter subfamilies invariant under permutations of factors. The latter allows to apply the Picard-Fuchs operators of order 4 to the calculation of predictions for numbers of rational curves on Calabi-Yau 3-folds with $h^{1,1} > 1$.

8.1 Calabi-Yau 3-folds in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$

We consider the diagonal subfamily of Kähler classes on Calabi-Yau hypersurfaces of degree (1, 1, 1, 1) in $(\mathbf{P}^1)^4$. Repeating the same procedure as for hypersurfaces of degree (3, 3) in $\mathbf{P}^2 \times \mathbf{P}^2$, we obtain:

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k_1+k_2+k_3+k_4=n} \frac{(2k_1+2k_2+2k_3+2k_4)!}{(k_1!)^2(k_2!)^2(k_3!)^2(k_4!)^2} \right) z^n$
P	$\Theta^4 - 4z(5\Theta^2 + 5\Theta + 2)(2\Theta + 1) + 64z^2(2\Theta + 3)(2\Theta + 1)(2\Theta + 2)^2$
$K_{z}^{(3)}$	$\frac{48}{(1-64z)(1-16z)}$
$K_{q}^{(3)}$	$48 + 192q + 7872q^2 + 278400q^3 + 9445056q^4 + 315072192q^5 + O(q^6)$
n_i	$n_1 = 192, n_2 = 960, n_3 = 10304, n_4 = 147456, n_5 = 2520576$

Proposition 8.1.1 The number of lines on a generic Calabi-Yau hypersurface in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ relative to the (1, 1, 1, 1)-polarization is equal to 192.

Proof. Let f be the polynomial of degree (2, 2, 2, 2) defining a Calabi-Yau hypersurface V in $(\mathbf{P}^1)^4$. If V contains a (0, 0, 0, 1)-curve whose projection on the product of first three \mathbf{P}^1 is a point (p_1, p_2, p_3) , then all three coefficients of the binary quadric obtained from f by substitution of (p_1, p_2, p_3) must vanish. Hence, the number of (0, 0, 0, 1) curves on V equals the intersection number of 3 hypersurfaces of degree (2, 2, 2) in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. This number is 48. By symmetry, the total amount of lines on V is $4 \times 48 = 192$.

Proposition 8.1.2 The number of conics on a generic Calabi-Yau hypersurface in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with respect to the (1, 1, 1, 1)-polarization is equal to 960.

Proof. By symmetry, it is sufficient to compute the number of rational curves of type (0, 0, 1, 1). Let M be the product of first two \mathbf{P}^1 in $(\mathbf{P}^1)^4$. Then we obtain the natural embedding

$$\phi : M \hookrightarrow \mathbf{P}^8 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2, 2)).$$

On the other hand, the points on M corresponding to projections of (0, 0, 1, 1)-curves on V are intersections of $\phi(M)$ with the 6-dimensional subvariety $\Lambda \subset \mathbf{P}^8$ which is the image of the morphism

$$\phi' : \mathbf{P}(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1))^2 = \mathbf{P}^3 \times \mathbf{P}^3 \to \mathbf{P}^8 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(2,2)).$$

The image $\phi(M)$ has degree 8. On the other hand, ϕ has degree two onto its image. Hence, the subvariety Λ has degree 10. Hence, we obtain $8 \times 20 = 160$ points on M. There are 6 possibilities for the choice of the type of conics. Thus, the total amount of conics is $6 \times 160 = 960$.

8.2 Complete intersections of three hypersurfaces in $P^2 \times P^2 \times P^2$

We consider two examples of 3-dimensional complete intersections with trivial canonical class in $(\mathbf{P}^2)^3$.

Calabi-Yau complete intersections of 3 hypersurfaces of degree (1, 1, 1):

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m+l=n} \frac{((k+m+l)!)^3}{(k!)^3 (m!)^3 (l!)^3} \right) z^n$
\mathcal{P}	$25\Theta^{4} - 15z(5 + 30\Theta + 72\Theta^{2} + 84\Theta^{3} + 51\Theta^{4}) +6z^{2}(15 + 155\Theta + 541\Theta^{2} + 828\Theta^{3} + 531\Theta^{4}) -54z^{3}(1170 + 3795\Theta + 4399\Theta^{2} + 2160\Theta^{3} + 423\Theta^{4}) +243z^{4}(402 + 1586\Theta + 2270\Theta^{2} + 1386\Theta^{3} + 279\Theta^{4}) - 59049z^{5}(\Theta + 1)^{4}$
$K_{z}^{(3)}$	$\frac{90 + 162z}{(27z - 1)(27z^2 + 1)}$
$K_{q}^{(3)}$	$90 + 108q + 2916q^2 + 57456q^3 + 834084q^4 + 13743108q^5 + O(q^6)$
n_i	$n_1 = 108, n_2 = 351, n_3 = 2124, n_4 = 12987, n_5 = 109944$

Proposition 8.2.1 A generic complete intersection of 3 hypersurfaces of degree (1, 1, 1)in $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$ contains 108 lines relative to the $\mathcal{O}(1, 1, 1)$ -polarization. **Proof.** Let V be the complete intersection of three generic hypersurfaces V_1 , V_2 , V_3 in $M_1 \times M_2 \times M_3$, where $M_i \cong \mathbf{P}^2$.

By symmetry, it is sufficient to consider lines having the class (0, 0, 1) whose projections on $M_1 \times M_2$ are points. There is the morphism

$$\phi : M_1 \times M_2 \to \mathbf{P}^8 = \mathbf{P}(E)$$

where E is the space of all 3×3 -matrices. By definition, ϕ maps a point $(p_1, p_2) \in M_1 \times M_2$ to the matrix of coefficients of there linear forms obtained from the substitution of p_1 and p_2 in the equations of V_1, V_2 , and V_3 . The morphism ϕ is the Segre embedding and its image has degree 6. On the other hand, if a point $(p_1, p_2) \in M_1 \times M_2$ is a projection of a (0, 0, 1)-curve on V, then the image $\phi(p_1, p_2)$ must correspond to a matrix of rank 1 in E. Thus, the number of (0, 0, 1)-curves equals $6 \times 6 = 36$, the intersection number of two Segre subvarieties in \mathbf{P}^8 . So the number of lines on V is $3 \times 36 = 108$.

Abelian 3-folds: The complete intersection of three hypersurfaces of degrees (3, 0, 0), (0, 3, 0), and (0, 0, 3) are are abelian 3-folds constructed by taking products of 3 elliptic cubic curves in \mathbf{P}^2 . Although abelian varieities are not Calabi-Yau manifolds from view point of algebraic geometers, these manifolds also present interest for physicists.

$F_0(z)$	$\sum_{p+q+r=n} \left(\frac{((3p)!)^3 ((3q)!)^3 ((3r)!)^3}{(p!)^3 (q!)^3 (r!)^3} \right) z^n$
P	$\Theta^{4} - 3z(6 + 29\Theta + 56\Theta^{2} + 54\Theta^{3} + 27\Theta^{4}) +81z^{2}(27\Theta^{2} + 54\Theta + 40)(\Theta + 1)^{2} -2187z^{3}(3\Theta + 5)(3\Theta + 4)(\Theta + 2)(\Theta + 1)$
$K_{q}^{(3)}$	162

Thus, we obtain that all Gromov-Witten invariants for the abelian 3-folds are zero which agrees with the fact that there are no rational curves on abelian varieties.

8.3 Calabi-Yau 3-folds in $P^3 \times P^3$

Complete intersections of a hypersurface of degree (2,2) and 2 copies of hypersurfaces of degree (1,1):

$$F_{0}(z) \qquad \sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{(2k+2m)!((k+m)!)^{2}}{(k!)^{4}(m!)^{4}} \right) z^{n}$$

$$\mathcal{P} \quad \Theta^{4} - 4z(3\Theta + \Theta + 1)(2\Theta + 1)^{2} - 4z^{2}(4\Theta + 5)(4\Theta + 6)(4\Theta + 2)(4\Theta + 3))$$

$$K_{z}^{(3)} \qquad \frac{40}{(1+16z)(1-64z)}$$

$$K_{q}^{(3)} \quad 40 + 160q + 12640q^{2} + 393280q^{3} + 17420640q^{4} + 662416160q^{5} + O(q^{6})$$

$$n_{i} \qquad n_{1} = 160, n_{2} = 1560, n_{3} = 14560, n_{4} = 272000, n_{5} = 5299328$$

Proposition 8.3.1 The number of lines on a generic complete intersections of a hypersurface of degree (2, 2) and 2 copies of hypersurfaces of degree (1, 1) is equal to 160.

Proof. Let $W = \operatorname{Gr}(2,4) \times \mathbf{P}^3$ be the 7-dimensional variety parametrizing all (1,0)lines on $\mathbf{P}^3 \times \mathbf{P}^3$. Let \mathcal{E} be the tautological rank-2 locally free sheaf on $\operatorname{Gr}(2,4)$. We put $c_1(\mathcal{E}) = c_1, c_2(\mathcal{E}) = c_2$, and h be the first Chern class of the ample generator H of $\operatorname{Pic}(\mathbf{P}^3)$. Let $S^2(\mathcal{E})$ be the 2-nd symmetric power of \mathcal{E} . By standard arguments, we obtain:

Lemma 8.3.2 The Chern classes c_1 , c_2 generate the cohomology ring of Gr(2,4). The elements 1, c_1 , c_2 , c_1^2 , c_1c_2 , $c_1^2c_2$ form a **Z**-basis of $H^*(Gr(2,4), \mathbf{Z})$, and one has the following

$$c_1^4 = c_2^2 = c_1^2, \ c_1^3 = 2c_1c_2;$$

$$c_1(S^2(\mathcal{E})) = 3c_1, \ c_2(S^2(\mathcal{E})) = 2c_1^2 + 4c_2, \ c_3(S^2(\mathcal{E})) = 4c_1c_2.$$

Moreover, for any invertible sheaf \mathcal{L} on \mathbf{P}^3 , one has

$$c_1(S^2(\mathcal{E}) \otimes \mathcal{L}) = 3c_1 + 3c_1(\mathcal{L}),$$

$$c_2(S^2(\mathcal{E}) \otimes \mathcal{L}) = 2c_1^2 + 4c_2^2 + 2c_1(\mathcal{L})(3c_1) + 3c_1^2(\mathcal{L}),$$

$$c_3(S^2(\mathcal{E}) \otimes \mathcal{L}) = 4c_1c_2 + c_1(\mathcal{L})(2c_1^2 + 4c_2) + c_1^2(\mathcal{L})(3c_1) + c_1^3(\mathcal{L}),$$

Then the number of (1, 0)-lines equals the following product in the cohomology ring of W:

$$c_2(\mathcal{E} \otimes \mathcal{O}(H)) \cdot c_2(\mathcal{E} \otimes \mathcal{O}(H)) \cdot c_3(S^2(\mathcal{E}) \otimes \mathcal{O}(2H))$$

$$= (h^{2} + c_{1}h + c_{2})^{2}(8h^{3} + 3c_{1} \cdot 4h^{2} + 2(c_{1}^{2} + 2c_{2}) \cdot 2h + 4c_{1}c_{2})$$

$$= (8c_{1}^{2}c_{2}h^{3} + 4(c_{1}^{2} + 2c_{2})^{2}h^{3} + 24c_{1}^{2}c_{2}h^{3} + 8c_{2}^{2}h^{3})$$

$$= (8 + 4 \times 10 + 24 + 8)c_{1}^{2}c_{2}h^{3} = 80c_{1}^{2}c_{2}h^{3}.$$

Thus, the number of (1,0)-lines is 80. By symmetry, the total amount of lines is 160. \Box

Complete intersections of hypersurfaces of degrees (1,1), (1,2) and (2,1):

Proposition 8.3.3 The number of lines on a generic complete intersections of hypersurfaces of degrees (2, 1), (1, 2), and (1, 1) is equal to 160.

Proof. We use the same notations as in 8.3.1. The number of (1, 0)-lines equals the the following product in the cohomology ring of W:

$$c_{2}(\mathcal{E} \otimes \mathcal{O}(H)) \cdot c_{2}(\mathcal{E} \otimes \mathcal{O}(2H)) \cdot c_{3}(S^{2}(\mathcal{E}) \otimes \mathcal{O}(H))$$

= $(h^{2} + c_{1}h + c_{2}) \cdot (4h^{2} + 2c_{1}h + c_{2}) \cdot (h^{3} + 3c_{1}h^{2} + 2(c_{1}^{2} + 2c_{2})h + 4c_{1}c_{2})$
= $(24c_{1}^{2}c_{2} + 2(5c_{2} + 2c_{1}^{2})(2c_{2} + c_{1}^{2}) + 9c_{1}^{2}c_{2} + c_{2}^{2})h^{3}$
= $(24 + 2(5 + 10 + 4 + 4) + 9 + 1)c_{1}^{2}c_{2}h^{3} = 80c_{1}^{2}c_{2}h^{3}.$

Thus, the number of (1,0)-lines equals 80. By symmetry, the total amount of lines is 160. \square

Hypersurfaces in product of two Del Pezzo surfaces of degree 3:

A Calabi-Yau hypersurface in product of two Del Pezzo surfaces of degree 3 is a complete intersections of (1, 1), (3, 0) and (0, 3)-hypersurfaces in $\mathbf{P}^3 \times \mathbf{P}^3$.

$$\begin{array}{|c|c|c|c|c|} F_0(z) & \sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{(3k)!(3m)!(k+m)!}{(k!)^4(m!)^4} \right) z^n \\ \mathcal{P} & \bigoplus_{\substack{+9z^2(304+3z(4+23\Theta+53\Theta^2+60\Theta^3+48\Theta^4)\\ +9z^2(304+1344\Theta+2319\Theta^2+1980\Theta^3+873\Theta^4)\\ -162z^3(800+3348\Theta+5259\Theta^2+3888\Theta^3+1269\Theta^4)\\ +2916z^4(688+2952\Theta+4653\Theta^2+3240\Theta^3+891\Theta^4\\ -1417176z^5(3\Theta+4)(3\Theta+2)(\Theta+1)^2 \\ \hline K_q^{(3)} & \frac{54-972z}{(1-54z)(1-27z)^2} \\ \hline K_q^{(3)} & 54+162q+7290q^2+119232q^3+3045114q^4+79845912q^5+O(q^6)\\ \hline n_i & n_1=162, n_2=891, n_3=4410, n_4=47466, n_5=638766 \\ \end{array}$$

Proposition 8.3.4 Let S_1 , S_2 be two Del Pezzo surfaces of degree 3. Then the number of lines on a generic Calabi-Yau hypersurface V in $S_1 \times S_2$ is 162.

Proof. If C is a line of type (1,0) on $S_1 \times S_2$, then $\pi_1(C)$ is one of 27 lines on S_1 , and $\pi_2(C)$ is a point on S_2 . Let $\mathcal{O}_{S_i}(-K)$ denotes the anticanonical bundle over S_i . Then the zero set of a generic global section s of $\pi_1^*\mathcal{O}_{S_1}(-K) \otimes \pi_2^*\mathcal{O}_{S_2}(-K)$ defines a morphism

$$\phi : S_2 \to \mathbf{P}^3 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

On the other hand, for any line $L \in S_1$, one has the linear embedding

$$\phi'$$
: $\mathbf{P}(\mathcal{O}_{S_1}(-K-L)) \cong \mathbf{P}^1 \hookrightarrow \mathbf{P}^3 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$

The intersection number of Im ϕ and Im ϕ' in \mathbf{P}^3 equals 3, i.e., one has exactly 3 lines C on a generic V such that $\pi_1(C) = L$ and $\pi_2(C)$ is a point on S_2 . Thus, there are $3 \times 27 = 81$ lines of type (1,0) on V. By symmetry, the total amount of lines is 162.

Proposition 8.3.5 Let S_1 , S_2 be two Del Pezzo surfaces of degree 3. Then the number of conics on a generic Calabi-Yau hypersurface V in $S_1 \times S_2$ is 891.

Proof. If C is a conic of type (1,1) on $S_1 \times S_2$, then $L_1 = \pi_1(C)$ is one of 27 lines on S_1 , and $L_2 = \pi_2(C)$ is one of 27 lines on S_2 . On the other hand, for any pair of lines $L_1 \in S_1, L_2 \in S_2$, the intersection of the product $L_1 \times L_2 \subset S_1 \times S_2$ with V is a conic of type (1,1). So we obtain $27 \times 27 = 729$ conics of type (1,1) on V. On the other hand, the number of (2,0) and (0,2) conics is obviously equals to the number of (1,0) and (0,1)lines. Thus, the total number of conics is equal to 729 + 162 = 891.

8.4 Calabi-Yau 3-folds in $P^4 \times P^4$

Complete intersection of hypersuraces of degrees (2,0), (0,2), and 3 copies of hypersurfaces of degree (1,1):

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{((k+m)!)^3 (2k)! (2m)!}{(k!)^5 (m!)^5} \right) z^n$
P	$25\Theta^{4} - 20z(5 + 30\Theta + 72\Theta^{2} + 84\Theta^{3} + 36\Theta^{4}) -16z^{2}(-35 - 70\Theta + 71\Theta^{2} + 268\Theta^{3} + 181\Theta^{4}) +256z^{3}(\Theta + 1)(165 + 375\Theta + 248\Theta^{2} + 37\Theta^{3}) +1024z^{4}(59 + 232\Theta + 331\Theta^{2} + 198\Theta^{3} + 39\Theta^{4}) + 32768z^{5}(\Theta + 1)^{4}$
$K_{z}^{(3)}$	$\frac{80 + 128z}{(1+4z)(1-4z)(1-32z)}$
$K_{q}^{(3)}$	$80 + 128q + 3776q^2 + 65792q^3 + 1299136q^4 + 23104128q^5 + O(q^6)$
n_i	$n_1 = 128, n_2 = 456, n_3 = 2432, n_4 = 20240, n_5 = 184832$

Proposition 8.4.1 The number of lines on a generic complete intersections of hypersuraces of degrees (2,0), (0,2), and 3 copies of hypersurfaces of degree (1,1) is equal to 128.

Proof. Let $W = \operatorname{Gr}(2,5) \times \mathbf{P}^4$ be the 10-dimensional variety parametrizing all (1,0)lines on $\mathbf{P}^4 \times \mathbf{P}^4$. Let \mathcal{E} be the tautological rank-2 locally free sheaf on $\operatorname{Gr}(2,5)$. We put $c_1(\mathcal{E}) = c_1, c_2(\mathcal{E}) = c_2$, and h be the first Chern class of the ample generator H of $\operatorname{Pic}(\mathbf{P}^4)$. Let $S^2(\mathcal{E})$ be the 2-nd symmetric power of \mathcal{E} . Again, by standard arguments, we obtain:

Lemma 8.4.2 The Chern classes c_1 , c_2 generate the cohomology ring of Gr(2,5). The elements 1, c_1 , c_2 , c_1^2 , c_1c_2 , c_1^3 , $c_1^2c_2$, c_1^4 , $c_1c_2^2$, c_2^3 form a **Z**-basis of $H^*(Gr(2,5), \mathbf{Z})$, and satisfy the following relations

$$c_1^4 c_2 = 2c_1^2 c_2^2 = 2c_2^3, \ c_1^5 = 5c_1 c_2^2, \ c_1^6 = 5c_1^2 c_2^2 = 5c_2^3, \ c_1^3 c_2 = 2c_1 c_2^2.$$

Then the number of (1, 0)-lines equals the following product in the cohomology ring of W:

$$c_1(\mathcal{O}(H) \cdot (c_2(S^2(\mathcal{E})))^3 \cdot c_3(S^2(\mathcal{E})))$$

= $(2h) \cdot (h^2 + c_1h + c_2)^3 \cdot (4c_1c_2) = 64c_1^2c_2^2h^4$

Thus, the number of (1, 0)-lines is 64. By symmetry, the total amount of lines is 128. \Box

Complete intersection of 5 hypersurfaces of degree (1,1):

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{((k+m)!)^5}{(k!)^5 (m!)^5} \right) z^n$
$\mathcal P$	$\begin{array}{r} 49\Theta^4 - 7z(14 + 91\Theta + 234\Theta^2 + 286\Theta^3 + 155\Theta^4) \\ -z^2(15736 + 66094\Theta + 102261\Theta^2 + 680044\Theta^3 + 16105\Theta^4) \\ +8z^3(476 + 3759\Theta + 9071\Theta^2 + 8589\Theta^3 + 2625\Theta^4) \\ -16z^4(184 + 806\Theta + 1439\Theta^2 + 1266\Theta^3 + 465\Theta^4) + 512z^5(\Theta + 1)^4 \end{array}$
$K_{z}^{(3)}$	$\frac{70 - 40z}{(32z - 1)(z^2 - 11z - 1)}$
$K_{q}^{(3)}$	$K_q(q) = 70 + 100q + 5300q^2 + 79750q^3 + 1966900q^4 + 37143850q^5 + O(q^6)$
n_i	$n_1 = 100, n_2 = 650, n_3 = 2950, n_4 = 30650, n_5 = 297150$

Proposition 8.4.3 A generic complete intersection of generic 5 hypersurfaces of degree (1, 1) in $\mathbf{P}^4 \times \mathbf{P}^4$ contains 100 lines.

Proof. We give below two different proofs of the statement.

I: We keep the notation from the proof of 8.4.1. Then the number of (1, 0)-lines equals the following product in the cohomology ring of W:

$$(c_2(S^2(\mathcal{E}))^5 = (h^2 + c_1h + c_2)^5$$
$$= (c_1h + c_2)65 + 5(c_1h + c_2)64h^2 + 10(c_1h + c_2)^3h^4 = 5c_1^4c_2h^4 + 5\binom{4}{2}c_1^2c_2^2h^4 + 10c_2^3h^4$$
$$= (5 \times 2 + 5 \times 6 + 10)c_1^4c_2h^4 = 50c_1^2c_2^2h^4.$$

Thus, the number of (1, 0)-lines is 50. By symmetry, the total amount of lines is 100.

II: Let $M = M_1 \times M_2$ where $M_i \cong \mathbf{P}^4$ (i = 1, 2). By symmetry, we consider only lines of type (0, 1) whose projections on M_1 are points. The substitution of a point $p \in M_1$ in the equations of the hypersurfaces $H_1, \ldots, H_5 \subset M$ gives 5 linear forms f_1, \ldots, f_5 in homogeneous coordinates on M_2 . A point $p \in M_1$ is a projection of a (0, 1)-line on $H_1 \cap \cdots \cap H_5$ if the system of linear forms has rank 3. The space of 5 copies of linear forms can be identified with the space L of matrices 5×5 . We are interested in the determinantal subvariety D in \mathbf{P}^{24} consisting of matrices of rank ≤ 3 . The subvariety D has the codimension 4, the ideal of D is generated by all 4×4 minors. Using the free graded resolution of the homogeneous coordinate ring of D as a module over the homogeneous coordinate ring of \mathbf{P}^{24} , we can compute the degree of D which is equal to 50 (The Hilbert-Poincare series equals $(1+4t+10t^2+20t^3+10t^4+4t^5+t^6)/(1-t)^{21}$). On the other hand, the equatons of generic hypersurfaces H_1, \ldots, H_5 define a generic embedding

$$\mathbf{P}^4 \hookrightarrow \mathbf{P}^{24}$$

of \mathbf{P}^4 as a linear subspace. So the number of lines of type (0,1) on a generic complete intersection is 50. Thus, the total number of lines is 100.

Hypersurfaces in product of two Del Pezzo surfaces of degree 4:

A Calabi-Yau hypersurface in the product of two Del Pezzo surfaces of degree 4 is a complete intersection of 5 hypersurfaces in $\mathbf{P}^4 \times \mathbf{P}^4$: two copies of type (2,0), two copies of type (0,2), and one copy of type (1,1).

$F_0(z)$	$\sum_{n=0}^{\infty} \left(\sum_{k+m=n} \frac{((2k)!)^2 ((2m)!)^2 (k+m)!}{(k!)^5 (m!)^5} \right) z^n$
$\mathcal P$	$\begin{array}{r} 9\Theta^4 - 4z(6 + 33\Theta + 73\Theta^2 + 80\Theta^3 + 64\Theta^4) \\ + 128z^2(75 + 315\Theta + 527\Theta^2 + 440\Theta^3 + 194\Theta^4) \\ - 4096z^3(66 + 261\Theta + 397\Theta^2 + 288\Theta^3 + 94\Theta^4) \\ + 131072z^4(19 + 77\Theta + 117\Theta^2 + 80\Theta^3 + 22\Theta^4) - 8388608z^5(\Theta + 1)^4 \end{array}$
$K_{z}^{(3)}$	$\frac{96 - 1024z}{(1 - 32z)(1 - 16z)^2}$
$K_{q}^{(3)}$	$96 + 128q + 3456q^2 + 38144q^3 + 572800q^4 + 9344128q^5 + O(q^6)$
n_i	$n_1 = 128, n_2 = 416, n_3 = 1408, n_4 = 8896, n_5 = 74752$

Proposition 8.4.4 Let S_1 , S_2 be two Del Pezzo surfaces of degree 4. Then the number of lines on a generic Calabi-Yau hypersurface V in $S_1 \times S_2$ is 128.

Proof. If C is a line of type (1,0) on $S_1 \times S_2$, then $\pi_1(C)$ is one of 16 lines on S_1 , and $\pi_2(C)$ is a point on S_2 . Let $\mathcal{O}_{S_i}(-K)$ denotes the anticanonical bundle over S_i . Then the zero set of a generic global section s of $\pi_1^*\mathcal{O}_{S_1}(-K) \otimes \pi_2^*\mathcal{O}_{S_2}(-K)$ defines a morphism

$$\phi : S_2 \to \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

On the other hand, for any line $L \in S_1$, one has the linear embedding

$$\phi'$$
: $\mathbf{P}(\mathcal{O}_{S_1}(-K-L)) \cong \mathbf{P}^2 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$

The intersection number of Im ϕ and Im ϕ' in \mathbf{P}^3 equals 4, i.e., one has exactly 4 lines C on a generic V such that $\pi_1(C) = L$ and $\pi_2(C)$ is a point on S_2 . Thus, there are $4 \times 16 = 64$ lines of type (1,0) on V. By symmetry, the total amount of lines is 128.

Proposition 8.4.5 Let S_1 , S_2 be two Del Pezzo surfaces of degree 3. Then the number of conics on a generic Calabi-Yau hypersurface V in $S_1 \times S_2$ is 416.

Proof. If C is a conic of type (1,1) on $S_1 \times S_2$, then $L_1 = \pi_1(C)$ is one of 16 lines on S_1 , and $L_2 = \pi_2(C)$ is one of 16 lines on S_2 . On the other hand, for any pair of lines $L_1 \in S_1, L_2 \in S_2$, the intersection of the product $L_1 \times L_2 \subset S_1 \times S_2$ with V is a conic of type (1,1). So we obtain $16 \times 16 = 256$ conics of type (1,1) on V.

In order to compute the number of (2,0)-conics, we notice that S_1 has exactly 10 of conic bundle structures. Moreover, these conic bundle structures can be divided into 5 pairs such that the union of degenerate fibers of each pair is the set of all 16 lines on S_1 . A generic global section s of $\pi_1^* \mathcal{O}_{S_1}(-K) \otimes \pi_2^* \mathcal{O}_{S_2}(-K)$ defines the anticanonical embedding

$$\phi : S_2 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

On the other hand, the points $p \in S_2$ such that $\phi(p)$ splits into the union of two conics $C_1 \cup C_2$ are exactly intersection points of $\phi(S_2)$ and the image of the embedding

$$\phi' : \mathbf{P}(\mathcal{O}_{S_1}(C_1) \times \mathcal{O}_{S_1}(C_2) \cong \mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^4 = \mathbf{P}(\mathcal{O}_{S_1}(-K)).$$

Since the image of ϕ' has degree 2, we obtain 8 points $p \in S_2$. Each such a point yields 2 conics on $\pi_2^{-1}(p)$. Therefore, for each of 5 pairs of conic bundle structures we have 16 (2, 0)-conics.

Thus, the total number of conics is equal to $256 + 2 \times 80 = 416$.

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